Graded Dominance

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1 Introduction

The relation of dominance between aggregation operators has recently been studied quite intensively [9, 10]. We propose to study its 'graded' generalization in the foundational framework of higher-order fuzzy logic, also known as Fuzzy Class Theory (FCT) introduced in [1]. FCT is specially designed to allow a quick and sound development of graded, lattice-valued generalizations of the notions of traditional 'fuzzy mathematics' and is a backbone of a broader program of logic-based foundations for fuzzy mathematics, described in [2].

This short abstract is to be understood as just a 'teaser' of the broad and potentially very interesting area of graded dominance. We sketch basic definitions and properties related to this notion and present a few examples of results in the area of equivalence and order relations (in particular, we show interesting graded generalization of basic results from [6]). Also some of our theorems are, for expository purposes, stated in a less general form here and can be further generalized substantively.

In this paper, we work in Fuzzy Class Theory over the logic MTL_Δ of all left-continuous t-norms [7]. The apparatus of FCT and its standard notation is explained in detail in the primer [3], which is freely available online. Furthermore we use $X \sqsubseteq Y$ for $\Delta(X \subseteq Y)$.

2 Inner Truth Values and Truth-Value Operators

An important feature of FCT is the absence of variables for truth values. However, many theorems of traditional fuzzy mathematics do speak about truth values or quantify over operators on truth values like aggregation operators, copulas, t-norms, etc. In order to be able to speak of truth values within FCT, truth values need be *internalized* in the theory. This is done in [4] by a rather standard technique, by representing truth values by subclasses of a crisp singleton.[4] Thus we can assume that we do have variables $\alpha, \beta, \ldots$ for truth values in FCT; the class of the inner truth values is denoted by L.

Binary operators on truth values (including propositional connectives $\&, \neg, \ldots$) can then be regarded as functions $\mathbf{c}\colon \mathrm{L} \times \mathrm{L} \to \mathrm{L}$ or as fuzzy relations $\mathbf{c} \sqsubseteq \mathrm{L} \times \mathrm{L}$. Consequently, graded class relations can be applied to such operators, e.g., fuzzy inclusion $\mathbf{c} \subseteq \mathbf{d} \equiv (\forall \alpha, \beta)(\alpha\, \mathbf{c}\, \beta \to \alpha\, \mathbf{d}\, \beta)$. Many crisp classes of truth-value operators (e.g., t-norms, continuous t-norms, copulas, etc.) can be defined by formulae of FCT. The apparatus, however, enables also *partial* satisfaction of such conditions. In the

[4] Cf. [11] for an analogous construction in a set theory over a variant of Gödel logic. See [4] for details of the construction and certain metamathematical qualifications regarding the representation. Observe also a parallel with the power-object of 1 in topos theory.

following, we therefore give several *fuzzy* conditions on truth-value operators and use them as graded preconditions of theorems which need not be satisfied to the full degree. This yields a completely new *graded* theory of truth-value operators and allows non-trivial generalizations of well-known theorems on such operators, including their consequences for properties of fuzzy relations.

Definition 1. In FCT, we define the following graded properties of a truth-value operator $\mathbf{c} \sqsubseteq \mathrm{L} \times \mathrm{L}$:

$$\begin{aligned}
\mathrm{Com}(\mathbf{c}) &\equiv_{\mathrm{df}} (\forall \alpha, \beta)(\alpha\, \mathbf{c}\, \beta \rightarrow \beta\, \mathbf{c}\, \alpha) \\
\mathrm{Ass}(\mathbf{c}) &\equiv_{\mathrm{df}} (\forall \alpha, \beta, \gamma)((\alpha\, \mathbf{c}\, \beta)\, \mathbf{c}\, \gamma) \leftrightarrow (\alpha\, \mathbf{c}\, (\beta\, \mathbf{c}\, \alpha)) \\
\mathrm{MonL}(\mathbf{c}) &\equiv_{\mathrm{df}} (\forall \alpha, \beta, \gamma)(\Delta(\alpha \rightarrow \beta) \rightarrow (\alpha\, \mathbf{c}\, \gamma \rightarrow \beta\, \mathbf{c}\, \gamma)) \\
\mathrm{MonR}(\mathbf{c}) &\equiv_{\mathrm{df}} (\forall \alpha, \beta, \gamma)(\Delta(\alpha \rightarrow \beta) \rightarrow (\gamma\, \mathbf{c}\, \alpha \rightarrow \gamma\, \mathbf{c}\, \beta)) \\
\mathrm{UnL}(\mathbf{c}) &\equiv_{\mathrm{df}} (\forall \alpha)(1\, \mathbf{c}\, \alpha \leftrightarrow \alpha) \\
\mathrm{UnR}(\mathbf{c}) &\equiv_{\mathrm{df}} (\forall \alpha)(\alpha\, \mathbf{c}\, 1 \leftrightarrow \alpha)
\end{aligned}$$

For convenience, we also define

$$\begin{aligned}
\mathrm{Mon}(\mathbf{c}) &\equiv_{\mathrm{df}} \mathrm{MonL}(\mathbf{c})\ \&\ \mathrm{MonR}(\mathbf{c}) \\
\mathrm{wMon}(\mathbf{c}) &\equiv_{\mathrm{df}} \mathrm{MonL}(\mathbf{c}) \wedge \mathrm{MonR}(\mathbf{c})
\end{aligned}$$

and analogously for Un.

The following theorem provides us with samples of basic graded results.

Theorem 1. *FCT proves the following graded properties of truth-value operators:*

1. $\mathrm{Mon}(\mathbf{c})\ \&\ \mathrm{Un}(\mathbf{c}) \rightarrow (\mathbf{c} \subseteq \wedge)$
2. $\mathrm{wMon}(\mathbf{c})\ \&\ (\forall \alpha)(\alpha\, \mathbf{c}\, \alpha \leftrightarrow \alpha) \rightarrow (\wedge \subseteq \mathbf{c})$
3. $\mathrm{Mon}(\mathbf{c})\ \&\ \mathrm{Un}(\mathbf{c}) \rightarrow [(\alpha\, \mathbf{c}\, \alpha \leftrightarrow \alpha) \leftrightarrow (\forall \beta)((\alpha\, \mathbf{c}\, \beta) \leftrightarrow (\alpha \wedge \beta))]$

The three assertions above are generalizations of well-known basic properties of t-norms. Theorem 1.1 corresponds to the fact that the minimum is the greatest (so-called strongest) t-norm. Theorem 1.2 generalizes the basic fact that the minimum is the only idempotent t-norm, while 1.3 is a graded characterization of the idempotents of **c**. [8].

3 Graded Dominance

Definition 2. The graded relation $\ll$ of *dominance* between truth-value operators is defined as follows:

$$\mathbf{c} \ll \mathbf{d} \equiv_{\mathrm{df}} (\forall \alpha, \beta, \gamma, \delta)((\alpha\, \mathbf{d}\, \gamma)\, \mathbf{c}\, (\beta\, \mathbf{d}\, \delta) \rightarrow (\alpha\, \mathbf{c}\, \beta)\, \mathbf{d}\, (\gamma\, \mathbf{c}\, \delta))$$

Theorem 2. *FCT proves the following graded properties of dominance:*

1. $\Delta\mathrm{Com}(\mathbf{c})\ \&\ \mathrm{Ass}^4(\mathbf{c})\ \&\ \mathrm{Mon}(\mathbf{c}) \rightarrow (\mathbf{c} \ll \mathbf{c})$
2. $\mathrm{Un}(\mathbf{c})\ \&\ \mathrm{Un}(\mathbf{d})\ \&\ (\mathbf{c} \ll \mathbf{d}) \rightarrow (\mathbf{c} \subseteq \mathbf{d})$
3. $\Delta\mathrm{Com}(\mathbf{c})\ \&\ \mathrm{Ass}^4(\mathbf{c})\ \&\ \mathrm{Mon}^2(\mathbf{c})\ \&\ (\mathbf{d} \sqsubseteq \mathbf{c})\ \&\ (\mathbf{c} \subseteq \mathbf{d}) \rightarrow (\mathbf{c} \ll \mathbf{d})$
4. $\Delta\mathrm{Com}(\mathbf{d})\ \&\ \mathrm{Ass}^4(\mathbf{d})\ \&\ \mathrm{Mon}^2(\mathbf{d})\ \&\ (\mathbf{d} \sqsubseteq \mathbf{c})\ \&\ (\mathbf{c} \subseteq \mathbf{d}) \rightarrow (\mathbf{c} \ll \mathbf{d})$
5. $\mathrm{Mon}(\mathbf{c})\ \&\ (\& \ll \mathbf{c})\ \&\ ((\alpha \rightarrow \beta)\, \mathbf{c}\, (\gamma \rightarrow \delta)) \rightarrow ((\alpha\, \mathbf{c}\, \gamma) \rightarrow (\beta\, \mathbf{c}\, \delta))$
6. $\mathrm{Mon}(\mathbf{c})\ \&\ (\& \ll \mathbf{c})\ \&\ ((\alpha \leftrightarrow \beta)\, \mathbf{c}\, (\gamma \leftrightarrow \delta)) \rightarrow ((\alpha\, \mathbf{c}\, \gamma) \leftrightarrow (\beta\, \mathbf{c}\, \delta))$

Theorems 2.1 and 2.2 are generalizations of two basic facts, namely that every t-norm dominates itself and that dominance implies inclusion / pointwise order. Theorems 2.3 and 2.4 have no correspondences among known results; they provide us with bounds for the degree to which $(\mathbf{c} \ll \mathbf{d})$ holds, where the assumption $(\mathbf{d} \sqsubseteq \mathbf{c})$ & $(\mathbf{c} \subseteq \mathbf{d})$ would be obviously useless in the crisp non-graded framework (as it necessitates that $\mathbf{c}$ and $\mathbf{d}$ coincide anyway). Theorem 2.5 provides us with strengthened monotonicity of an aggregation operator $\mathbf{c}$ provided that $\mathbf{c}$ fulfills $\mathrm{Mon}(\mathbf{c})$ and dominates the conjunction of the underlying logic. Theorem 2.6 is then a kind of "Lipschitz property" of $\mathbf{c}$ (if we view $\leftrightarrow$ as a kind of generalized closeness measure).

Theorem 3. *FCT proves the following graded properties of dominance w.r.t.* $\wedge$*:*

1. $\mathrm{Mon}(\mathbf{c}) \to (\mathbf{c} \ll \wedge)$
2. $\Delta \mathrm{Mon}(\mathbf{c})\ \&\ \Delta \mathrm{Un}(\mathbf{c}) \to ((\wedge \ll \mathbf{c}) = (\wedge \subseteq \mathbf{c}))$
3. $\mathrm{wMon}^2(\mathbf{c}) \to ((\wedge \ll \mathbf{c}) \leftrightarrow (\forall \alpha, \beta)((\alpha\, \mathbf{c}\, 1) \wedge (1\, \mathbf{c}\, \beta) \leftrightarrow (\alpha\, \mathbf{c}\, \beta)))$

Theorem 3.1 is a graded generalization of the well-known fact that the minimum dominates any aggregation operator [10]. Theorem 3.2 demonstrates a rather surprising fact: that the degree to which a monotonic binary operation with neutral element 1 dominates the minimum is nothing else but the degree to which it is larger. Theorem 3.3 is an alternative characterization of operators dominating the minimum; for its non-graded version see [10, Prop. 5.1].

Example 1. Assertion 2. of Theorem 3 can easily be utilized to compute degrees to which standard t-norms on the unit interval dominate the minimum. It can be shown easily that

$$(\wedge \subseteq \mathbf{c}) = \inf_{x \in [0,1]} (x \Rightarrow \mathbf{c}(x,x))$$

holds, i.e. the largest "difference" of a t-norm $\mathbf{c}$ from the minimum can always be found on the diagonal. In standard Łukasiewicz logic, this is, for instance, 0.75 for the product t-norm and 0.5 for the Łukasiewicz t-norm itself. So we can infer that the product t-norm dominates the minimum with a degree of 0.75 (assuming that the underlying logic is standard Łukasiewicz!); with the same assumption, the Łukasiewicz t-norm dominates the minimum to a degree of 0.5.

4 Graded Dominance and Properties of Fuzzy Relations

The following theorems show the importance of graded dominance for graded properties of fuzzy relations. Theorem 4 is a graded generalization of the well-known theorem by De Baets and Mesiar that uses dominance to characterize preservation of transitivity by aggregation [6, Th. 2].

Theorem 4. *FCT proves:*

$$\mathrm{Mon}(\mathbf{c}) \to ((\forall E, F)(\Delta \mathrm{Trans}(E)\ \&\ \Delta \mathrm{Trans}(F) \to \mathrm{Trans}(\mathrm{Op}_{\mathbf{c}}(E,F)) \leftrightarrow (\& \ll \mathbf{c})))$$

where $\mathrm{Op}_{\mathbf{c}}$ *is the class operation given by* $\mathbf{c}$*, i.e.,* $\langle x, y \rangle \in \mathrm{Op}_{\mathbf{c}}(E,F) \equiv Exy\ \mathbf{c}\ Fxy$*.*

The following theorem provides us with results on the preservation of various properties by symmetrizations of fuzzy relations.

Theorem 5. *FCT proves the following properties of the symmetrization of relations:*

1. $\mathrm{Com}(\mathbf{c}) \to (\mathrm{Sym}(\mathrm{Op}_{\mathbf{c}}(R, R^{-1})))$
2. $(\& \subseteq \mathbf{c}) \,\&\, \mathrm{Refl}^2 R \to (\mathrm{Refl}(\mathrm{Op}_{\mathbf{c}}(R, R^{-1})))$
3. $(\& \subseteq \mathbf{c}) \to \mathrm{AntiSym}_{(\mathrm{Op}_{\mathbf{c}}(R,R^{-1}))} R$
4. $\mathrm{Mon}(\mathbf{c}) \,\&\, (\& \ll \mathbf{c}) \,\&\, \Delta \mathrm{Trans} R \to (\mathrm{Trans}(\mathrm{Op}_{\mathbf{c}}(R, R^{-1})))$

In the crisp case, the commutativity of an operator trivially implies the symmetry of symmetrizations by this operator. In the graded case, Theorem 5.1 above states that the degree to which a symmetrization is actually symmetric is bounded below by the degree to which the aggregation operator **c** is commutative. Theorems 5.2–4 are also well-known in the non-graded case [5, 6, 12]. Obviously, 5.4 is a simple corollary of Theorem 4.

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