Binary Ordering-Based Modifiers

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Abstract

In continuation of research work on the unary ordering-based modifiers *'at least'* and *'at most'*, this paper is concerned with the construction of a fuzzy concept of *'between'* operator, both in an inclusive and a non-inclusive setting. After motivating the practical need for such operators, we introduce the basic framework and investigate its most important properties.

Keywords: Fuzzy Orderings, Ordering-Based Modifiers.

1 Introduction

Fuzzy systems have always been regarded as appropriate methodologies for controlling complex systems and for carrying out complicated decision processes [17]. The compactness of rule bases, however, is still a crucial issue—the surveyability and interpretability of a rule base decreases with its number of rules. In particular, if rule bases are represented as complete tables, the number of rules grows exponentially with the number of variables. Therefore, techniques for reducing the number of rules in a rule base while still maintaining the system's behavior and improving surveyability and interpretability should receive special interest. In this paper, we deal with operators which are supposed to serve as a key to rule base reduction—*ordering-based modifiers.*

Almost all fuzzy systems involving numerical variables implicitly use orderings. It is almost standard to decompose the universe of a linearly ordered system variable into a certain number of fuzzy sets by means of the ordering of the universe—typically resulting in labels like *'small'*, *'medium'*, or *'large'*.

Let us consider a simple example. Suppose that we have a system with two real-valued input variables x_1, x_2 and a real-valued output variable y, where all domains are divided into five fuzzy sets with the linguistic labels *'Z'*, *'S'*, *'M'*, *'L'*, and *'V'* (standing for *'approx. zero'*, *'small'*, *'medium'*, *'large'*, and *'very large'*, respectively).

$x_1 \backslash x_2$	Z	S	M	L	V
Z	Z	S	M	L	V
S	S	S	M	L	L
M	S	M	L	M	M
L	S	S	M	M	S
V	Z	S	M	S	Z

It is easy to see that, in the above table, there are several adjacent rules having the same consequent value. Assuming that we had a unique and unambiguous computational methodology to compute *'at least A'*, *'at most A'*, or *'between A and B'*, it would be possible to group and replace such neighboring rules. For instance, the three rules

IF x_1 is *'S'* AND x_2 is *'Z'* THEN y is *'S'*
IF x_1 is *'M'* AND x_2 is *'Z'* THEN y is *'S'*
IF x_1 is *'L'* AND x_2 is *'Z'* THEN y is *'S'*

could be replaced by the following rule[1]

[1]It depends on the underlying inference scheme whether the result is actually the same; we leave this aspect aside for the present paper, since this is not its major concern.

(adopting an inclusive view on the adverb *'between'*):

IF x_1 is *'btw. S and L'* AND x_2 is *'Z'* THEN y is *'S'*

Of course, there is actually no need to do so in such a simple case. Anyway, grouping neighboring rules in such a way could help to reduce the size of larger high-dimensional rule bases considerably.

It is considered as another opportunity for reducing the size of a rule base to store only some representative rules and to "interpolate" between them [13], where, in this context, we understand interpolation as a computational method that is able to obtain a meaningful conclusion even if an observation does not match any antecedent in the rule base [12]. In any case, it is indispensable to have criteria for determining between which rules the interpolation should take place. Beside distance, orderings play a fundamental role in this selection. As an alternative to distance-based methods [13], it is possible to fill the gap between the antecedents of two rules using a non-inclusive concept of fuzzy *'between'*.

In [1, 2, 6], a basic framework for defining the unary modifiers ATL and ATM (short for *'at least'* and *'at most'*, respectively) by means of image operators of fuzzy orderings has been introduced. This general approach has the following advantages: it is applicable to any kind of fuzzy set, it can be used for any kind of fuzzy ordering without any restriction to linearly ordered or real-valued domains, and it even allows to take a domain-specific context of indistinguishability into account.

This paper is concerned with an extension of this framework by two binary ordering-based modifiers named BTW and SBT which both represent fuzzy *'between'* operators, where BTW stands for the inclusive and SBT ("strictly between") stands for the non-inclusive interpretation.

2 Preliminaries

Throughout the whole paper, we will not explicitly distinguish between fuzzy sets and their corresponding membership functions. Consequently, uppercase letters will be used for both synonymously. The set of all fuzzy sets on a domain X will be denoted with $\mathcal{F}(X)$. As usual, we call a fuzzy set A *normalized* if there exists an $x \in X$ such that $A(x) = 1$ holds.

In general, *triangular norms* [11], i.e. associative, commutative, and non-decreasing binary operations on the unit interval (i.e. a $[0,1]^2 \to [0,1]$ mappings) which have 1 as neutral element, will be considered as our standard models of logical conjunction. In this paper, assume that T denotes a left-continuous triangular norm, i.e. a t-norm whose partial mappings $T(x,.)$ and $T(.,x)$ are left-continuous.

Definition 1. Let T be a t-norm. The *T-intersection* of two fuzzy sets $A, B \in \mathcal{F}(X)$ is defined by means of the following membership function:

$$(A \cap_T B)(x) = T\big(A(x), B(x)\big)$$

For $T = \min$, we will simply use the notation $A \cap B$. Correspondingly, the max-union of two fuzzy sets $A, B \in \mathcal{F}(X)$ is defined as

$$(A \cup B)(x) = \max\big(A(x), B(x)\big).$$

So-called residual implications will be used as the concepts of logical implication [7, 8].

Definition 2. For any left-continuous t-norm T, the corresponding *residual implication* $\vec{T}$ is defined as

$$\vec{T}(x,y) = \sup\{u \in [0,1] \mid T(u,x) \leq y\}.$$

The residual implication can be used to define a logical negation which logically fits to the t-norm and its implication.

Definition 3. The *negation* corresponding to a left-continuous t-norm T is defined as

$$N_T(x) = \vec{T}(x,0).$$

Lemma 4. *N_T is a left-continuous non-increasing $[0,1] \to [0,1]$ mapping. Moreover, the so-called law of contraposition holds*

$$\vec{T}(x,y) \leq \vec{T}\big(N_T(y), N_T(x)\big)$$

which also implies

$$x \leq N_T\big(N_T(x)\big).$$

Note that the reverse inequality does not hold in general (unlike the Boolean case, where $p \Rightarrow q$ is equivalent to $\neg q \Rightarrow \neg p$).

Definition 5. The *T-complement* of a fuzzy set $A \in \mathcal{F}(X)$ is defined as

$$(\complement_T A)(x) = N_T\big(A(x)\big).$$

Lemma 6. *The following holds for all fuzzy sets $A, B \in \mathcal{F}(X)$:*

1. $A \cap_T \complement_T A = \emptyset$
2. $A \subseteq \complement_T \complement_T A$
3. $A \subseteq B$ *implies* $\complement_T A \supseteq \complement_T B$

Lemma 7. *As long as only* min*-intersections and* max*-unions are considered, the so-called De Morgan laws hold:*

$$\complement_T(A \cup B) = \big(\complement_T A\big) \cap \big(\complement_T B\big)$$
$$\complement_T(A \cap B) = \big(\complement_T A\big) \cup \big(\complement_T B\big)$$

As usual, we call a fuzzy set on a product space $X \times X$ *binary fuzzy relation.* The following two kinds of binary fuzzy relations will be essential.

Definition 8. A binary fuzzy relation E on a domain X is called *fuzzy equivalence relation* with respect to T, for brevity *T-equivalence*, if and only if the following three axioms are fulfilled for all $x, y, z \in X$:

1. Reflexivity: $E(x, x) = 1$
2. Symmetry: $E(x, y) = E(y, x)$
3. T-transitivity:

$$T\big(E(x, y), E(y, z)\big) \le E(x, z)$$

Definition 9. Let $L : X^2 \to [0, 1]$ be a T-transitive binary fuzzy relation. L is called *fuzzy ordering* with respect to T and a T-equivalence E, for brevity *T-E-ordering*, if and only if it additionally fulfills the following two axioms for all $x, y \in X$:

1. E-reflexivity: $E(x, y) \le L(x, y)$
2. T-E-antisymmetry:

$$T\big(L(x, y), L(y, x)\big) \le E(x, y)$$

A subclass, which will be of special importance in the following, are so-called direct fuzzifications.

Definition 10. A T-E-ordering L is called a *direct fuzzification* of a crisp ordering $\preceq$ if and only if it admits the following resolution:

$$L(x, y) = \begin{cases} 1 & \text{if } x \preceq y \\ E(x, y) & \text{otherwise} \end{cases}$$

It is worth to mention that there is a one-to-one correspondence between direct fuzzifications of crisp linear orderings and so-called fuzzy weak orderings, i.e. reflexive and T-transitive binary fuzzy relations which fulfill strong completeness (i.e., for all $x, y \in X$, $\max(L(x, y), L(y, x)) = 1$) [2, 4].

3 Unary Ordering-Based Modifiers

Throughout the remaining paper, assume that we are given a T-E-ordering L (for some T-equivalence E and a left-continuous t-norm T). Then the unary ordering-based modifiers ATL and ATM are defined as follows [1, 2]:

$$\mathrm{ATL}(A)(x) = \sup\{T\big(A(y), L(y, x)\big) \mid y \in X\}$$
$$\mathrm{ATM}(A)(x) = \sup\{T\big(A(y), L(x, y)\big) \mid y \in X\}$$

In the case that L coincides with a crisp ordering $\preceq$, we will explicitly indicate that by using the notations LTR and RTL (short for “left-to-right” and “right-to-left continuations”) instead of ATL and ATM, respectively. It is easy to verify that the following simplified representation holds in such a case:

$$\mathrm{LTR}(A)(x) = \sup\{A(y) \mid y \preceq x\}$$
$$\mathrm{RTL}(A)(x) = \sup\{A(y) \mid y \succeq x\}$$

Moreover, for a given fuzzy set A, LTR is the smallest superset of A with a non-decreasing membership function and RTL is the smallest superset of A with a non-increasing membership function. For convenience, let us use the notation EXT for the so-called *extensional hull* operator of the T-equivalence E:

$$\mathrm{EXT}(A)(x) = \sup\{T\big(A(y), E(y, x)\big) \mid y \in X\}$$

Note that, for an arbitrary fuzzy set A, $\mathrm{EXT}(A)$ is the smallest superset fulfilling the property

$$T\big(A(x), E(x,y)\big) \leq A(y)$$

for all $x, y \in X$. This property is usually called *extensionality* [9, 10, 14].

Theorem 11. [2] *If L is a direct fuzzification of some crisp ordering $\preceq$, the following equalities hold:*

$$\begin{aligned}
\mathrm{ATL}(A) &= \mathrm{EXT}(\mathrm{LTR}(A)) = \mathrm{LTR}(\mathrm{EXT}(A)) \\
&= \mathrm{EXT}(A) \cup \mathrm{LTR}(A) \\
\mathrm{ATM}(A) &= \mathrm{EXT}(\mathrm{RTL}(A)) = \mathrm{RTL}(\mathrm{EXT}(A)) \\
&= \mathrm{EXT}(A) \cup \mathrm{RTL}(A)
\end{aligned}$$

Moreover, $\mathrm{ATL}(A)$ *is the smallest fuzzy superset of A which is extensional and has a non-decreasing membership function. Analogously,* $\mathrm{ATM}(A)$ *is the smallest fuzzy superset of A which is extensional and has a non-increasing membership function.*

The notion of convexity and convex hulls will the essential in the following.

Definition 12. Provided that the domain X is equipped with some crisp ordering $\preceq$ (not necessarily linear), a fuzzy set $A \in \mathcal{F}(X)$ is called *convex* (compare with [15, 16]) if and only if, for all $x, y, z \in X$,

$$x \preceq y \preceq z \quad \text{implies} \quad A(y) \geq \min\big(A(x), A(z)\big).$$

Lemma 13. *Assume that $\preceq$ is an arbitrary, not necessarily linear ordering on a domain X. Then the fuzzy set*

$$\mathrm{CVX}(A) = \mathrm{LTR}(A) \cap \mathrm{RTL}(A)$$

is the smallest convex fuzzy superset of A.

Theorem 14. [2] *With the assumptions of Theorem 11 and the definition*

$$\mathrm{ECX}(A) = \mathrm{ATL}(A) \cap \mathrm{ATM}(A),$$

the following representation holds:

$$\begin{aligned}
\mathrm{ECX}(A) &= \mathrm{EXT}\big(\mathrm{CVX}(A)\big) = \mathrm{CVX}\big(\mathrm{EXT}(A)\big) \\
&= \mathrm{EXT}(A) \cup \mathrm{CVX}(A)
\end{aligned}$$

Furthermore, $\mathrm{ECX}(A)$ *is the smallest fuzzy superset of A which is extensional and convex.*

4 The Inclusive Operator

Finally, we can now define an operator representing an inclusive version of *'between'* with respect to a fuzzy ordering.

Definition 15. Given two fuzzy sets $A, B \in \mathcal{F}(X)$, the binary operator BTW is defined as

$$\mathrm{BTW}(A, B) = \mathrm{ECX}(A \cup B).$$

Note that it can easily be inferred from basic properties of ATL and ATM that the following alternative representation holds:

$$\begin{aligned}
\mathrm{BTW}(A, B) = \big(&\mathrm{ATL}(A) \cup \mathrm{ATL}(B)\big) \\
&\cap \big(\mathrm{ATM}(A) \cup \mathrm{ATM}(B)\big)
\end{aligned}$$

This representation is particularly helpful to prove the following basic properties of the BTW operator.

Proposition 16. *The following holds for all fuzzy sets $A, B \in \mathcal{F}(X)$:*

1. $\mathrm{BTW}(A, B) = \mathrm{BTW}(B, A)$
2. $A \subset \mathrm{BTW}(A, B)$
3. $\mathrm{BTW}(A, \emptyset) = \mathrm{BTW}(A, A) = \mathrm{ECX}(A)$
4. $\mathrm{BTW}(A, B)$ *is extensional*

If L is a direct fuzzification of a crisp ordering $\preceq$, then $\mathrm{BTW}(A, B)$ *is convex as well and* $\mathrm{BTW}(A, B)$ *is the smallest convex and extensional fuzzy set containing both A and B.*

It is, therefore, justified (in particular due to Point 2. above) to speak of an inclusive interpretation. Moreover, it is even possible to show that BTW is an associative operation; hence $(\mathcal{F}(X), \mathrm{BTW})$ is a commutative semigroup.

5 The Non-Inclusive Operator

Now let us study how a *'strictly between'* operator can be defined. It seems intuitively clear that *'strictly between A and B'* should be a subset of $\mathrm{BTW}(A, B)$ which should not include any relevant parts of A and B.

Definition 17. The *'strictly between'* operator is a binary connective on $\mathcal{F}(X)$ which is defined as

$$\begin{aligned}\mathrm{SBT}(A, B) = \mathrm{BTW}(A, B) \\ \cap \complement_T\big((\mathrm{ATL}(A) \cap \mathrm{ATL}(B)) \\ \cup (\mathrm{ATM}(A) \cap \mathrm{ATM}(B))\big).\end{aligned}$$

Proposition 18. *The following holds for all fuzzy sets $A, B \in \mathcal{F}(X)$:*

1. $\mathrm{SBT}(A, B) = \mathrm{SBT}(B, A)$
2. $\mathrm{SBT}(A, B) \subseteq \mathrm{BTW}(B, A)$
3. $\mathrm{SBT}(A, \emptyset) = A$
4. $\mathrm{SBT}(A, B)$ *is extensional*

If L is a direct fuzzification of a crisp ordering $\preceq$, $\mathrm{SBT}(A, B)$ is convex as well. Moreover, if we assume that A and B are normalized, the following holds:

5. $\mathrm{SBT}(A, A) = \emptyset$
6. $\mathrm{ECX}(A) \cap_T \mathrm{SBT}(A, B) = \emptyset$

Note that the last equality particularly implies

$$A \cap_T \mathrm{SBT}(A, B) = \emptyset$$

which justifies to speak of an non-inclusive concept.

6 Ordering Properties

Despite basic properties that have already been presented in the previous two sections, it remains to be clarified whether the results $\mathrm{BTW}(A, B)$ and $\mathrm{SBT}(A, B)$ obtained by the two operators are really *lying between* A and B. We will approach this question from an ordinal perspective. It is straightforward to define the following binary relation on $\mathcal{F}(X)$:

$$\begin{aligned}A \preceq_L B \text{ iff } \mathrm{ATL}(A) \supseteq \mathrm{ATL}(B) \text{ and} \\ \mathrm{ATM}(A) \subseteq \mathrm{ATM}(B)\end{aligned}$$

This relation is reflexive, transitive, and antisymmetric up to the following equivalence relation:

$$A \sim_L B \text{ if and only if } \mathrm{ECX}(A) = \mathrm{ECX}(B)$$

Moreover, if we restrict ourselves to fuzzy numbers and to the natural ordering of real numbers, it is relatively easy to see that $\preceq_L$ coincides with the interval ordering of fuzzy numbers induced by the extension principle. It is, therefore, justified to consider $\preceq_L$ as a meaningful general concept of ordering of fuzzy sets with respect to a given fuzzy ordering L [2, 3, 5].

The following theorem gives a clear justification that we may consider the definitions of the operators BTW and SBT as appropriate.

Theorem 19. *Suppose that we are given two normalized fuzzy sets $A, B \in \mathcal{F}(X)$ such that $A \preceq_L B$ holds. Then the following inequality holds:*

$$A \preceq_L \mathrm{BTW}(A, B) \preceq_L B$$

Now let us assume that L is strongly complete (therefore, a direct fuzzification of a crisp linear ordering $\preceq$) and that there exists a value $x \in X$ which separates A and B in the following way (for all $y, z \in X$):

$$\begin{aligned}A(y) > 0 \textit{ implies } y \prec x \textit{ and} \\ B(z) > 0 \textit{ implies } x \prec z\end{aligned}$$

Then the following inequality holds too:

$$A \preceq_L \mathrm{SBT}(A, B) \preceq_L B$$

7 Examples

In order to underline these rather abstract results with an example, let us consider two fuzzy subsets of the real numbers:

$$\begin{aligned}A(x) &= \max(1 - 3 \cdot |1 - x|, \\ &\qquad 0.7 - 2 \cdot |1.5 - x|, 0) \\ B(x) &= \max(1 - |4 - x|, 0)\end{aligned}$$

It is easy to see that both fuzzy sets are normalized; B is convex, while A is not convex.

The natural ordering of real numbers $\leq$ is a fuzzy ordering with respect to any t-norm and the crisp equality. If we take the Łukasiewicz t-norm $T_{\mathbf{L}}(x, y) = \max(x+y-1, 0)$, we obtain the fuzzy sets $\mathrm{BTW}(A, B)$ and $\mathrm{SBT}(A, B)$ as shown in Figure 1.

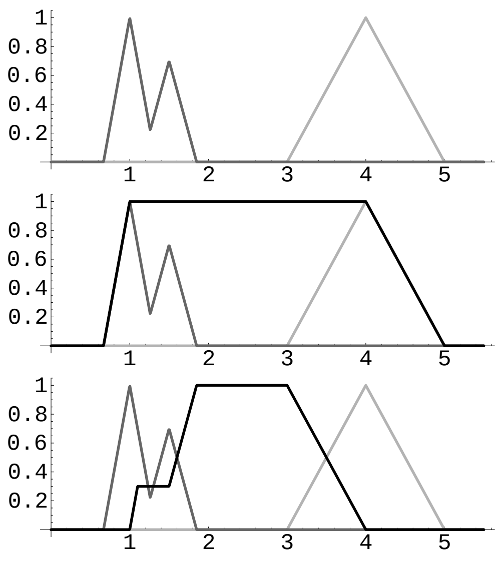

Figure 1: Two fuzzy sets A, B (top) and the results of $\mathrm{BTW}(A, B)$ (middle) and $\mathrm{SBT}(A, B)$ (bottom), using $T_{\mathbf{L}}$ and the crisp ordering of real numbers

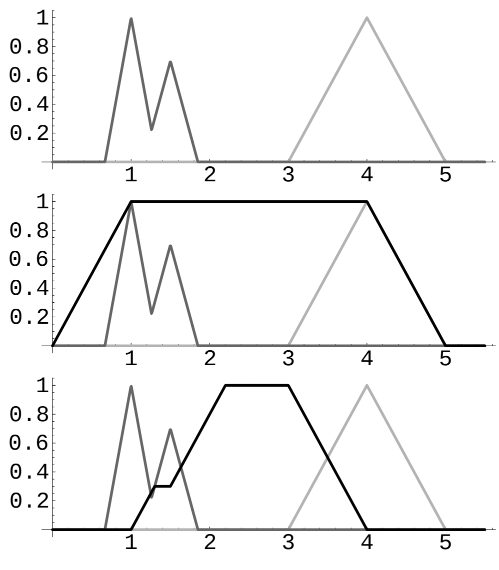

Figure 2: Two fuzzy sets A, B (top) and the results of $\mathrm{BTW}(A, B)$ (middle) and $\mathrm{SBT}(A, B)$ (bottom) with respect to a fuzzy ordering on $\mathbb{R}$

Now let us consider the following two fuzzy relations:

$$E(x, y) = \max(1 - |x - y|, 0)$$

$$L(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ E(x, y) & \text{otherwise} \end{cases}$$

One easily verifies that E is indeed a $T_{\mathbf{L}}$-equivalence on the real numbers and that L is a $T_{\mathbf{L}}$-E-ordering which directly fuzzifies the linear ordering of real numbers [2, 4]. Figure 2 shows the results of computing $\mathrm{BTW}(A, B)$ and $\mathrm{SBT}(A, B)$ for A and B from above. It is a routine matter to show that B is extensional and that A is not extensional. This means that A contains parts that are defined in an unnaturally precise way . Since the operators BTW and SBT have been designed to take the given context of indistinguishability into account, they try to remove all uncertainties arising from the non-extensionality of A. This is reflected in the fact that $\mathrm{BTW}(A, B)$ also contains some parts to the left of A that are potentially indistinguishable from A. In a dual way, $\mathrm{SBT}(A, B)$ does not include those parts to the right of A that are potentially indistinguishable from A.

8 Conclusion

This paper has been concerned with the definition of two binary ordering-based modifiers BTW and SBT. The operator BTW has been designed for computing the fuzzy set of all objects lying between two fuzzy sets including both boundaries. The purpose of SBT is to extract those objects which are lying strictly between two fuzzy sets—not including the two boundaries. We have shown several basic properties of the two operators and, from the viewpoint of orderings of fuzzy sets, that the two operators indeed yield meaningful results. Therefore, we conclude that the two operators are appropriate as modifiers for fuzzy systems applications and rule interpolation.

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