Fuzzy Orderings of Fuzzy Sets

Ulrich Bodenhofer

Software Competence Center Hagenberg A-4232 Hagenberg, Austria ulrich.bodenhofer@scch.at

Abstract. The purpose of this paper is to introduce a general framework for comparing fuzzy sets with respect to fuzzy orderings in a gradual way. This approach is applicable to fuzzy subsets of any kind of universe for which a fuzzy ordering, no matter whether linear or not, is known.

1 Introduction

Orderings and rankings are essential in any field related to decision making. Admitting vagueness or impreciseness naturally results in the need for specifying vague preferences in crisp domains, but also in the demand for techniques for deciding between fuzzy alternatives. It is, therefore, not surprising that orderings and rankings of fuzzy sets have become central objects of study in fuzzy decision analysis. Since the 1970s, a host of different methods for ordering or ranking fuzzy sets has been published (see [1–3] for detailed reviews). All of these approaches, however, have been based on strictly crisp comparisons of fuzzy sets. In the author's point of view, this is a serious restriction. Fuzzy sets have been introduced to model vagueness and impreciseness by admitting gradual membership . If a very small change in a degree of membership is sufficient to change an ordering or a ranking of fuzzy sets completely, the original motivation of fuzzy sets is disavowed.

This paper attempts at providing a way out of this pitfall. We start from a general fuzzy ordering [4–6] on the given domain (without any further restrictions) and define ordering-based modifiers [7,8]. Such ordering-based modifiers can be used to define a crisp ordering of fuzzy sets [9]. By employing an appropriate inclusion measure and a related similarity measure for fuzzy sets, this crisp ordering of fuzzy sets can be fuzzified.

2 Fuzzy Orderings

Throughout the whole paper, the symbols T, T', and \tilde{T} are supposed to denote left-continuous t-norms. As already mentioned, we do not want to restrict to specific domains (e.g. real numbers equipped with their natural ordering), but we want to start from a most general ordinal concept—a general fuzzy ordering on the domain X. This is mainly motivated by the fact that vague environments are equipped with a certain—implicit or explicit—context of indistinguishability [10]. For this purpose, similarity-based fuzzy orderings [4–6, 11], which are based on the idea to take an underlying context of indistinguishability into account, are perfectly suitable.

Definition 1. A binary fuzzy relation $E: X^2 \to [0, 1]$ is called *fuzzy equivalence* relation¹ with respect to T, for brevity T-equivalence, if and only if the following three axioms are fulfilled for all $x, y, z \in X$:

- 1. Reflexivity: E(x, x) = 1
- 2. Symmetry: E(x, y) = E(y, x)
- 3. T-transitivity: $T(E(x, y), E(y, z)) \leq E(x, z)$

Definition 2. Let $L: X^2 \to [0,1]$ be a binary fuzzy relation. L is called *fuzzy* ordering with respect to T and a T-equivalence $E: X^2 \to [0,1]$, for brevity T-*E*-ordering, if and only if it is T-transitive and additionally fulfills the following two axioms for all $x, y \in X$:

1. *E*-reflexivity: $E(x, y) \le L(x, y)$ 2. *T*-*E*-antisymmetry: $T(L(x, y), L(y, x)) \le E(x, y)$

3 Ordering-Based Modifiers

The modifiers 'at least' and 'at most' with respect to a fuzzy ordering L will be essential for these investigations. They can be defined by means of image operators [7, 8, 15].

Definition 3. Suppose that X is equipped with some T-E-ordering L. Then, for a fuzzy subset A of X, the fuzzy sets 'at least A' and 'at most A' (with respect to L), abbreviated ATL(A) and ATM(A), respectively, can be defined as follows:

 $ATL(A)(x) = \sup\{T(A(y), L(y, x)) \mid y \in X\}$ $ATM(A)(x) = \sup\{T(A(y), L(x, y)) \mid y \in X\}$

ATL(A) can be regarded as the smallest superset of A the membership function of which is non-decreasing with respect to the fuzzy ordering L, analogously for ATM(A). For detailed studies of the interpretation of these operators and their properties, we refer to the literature [7, 8, 11].

¹ Note that various diverging names for this class of fuzzy relations appear in literature, like similarity relations, indistinguishability operators, equality relations, and several more [10, 12–14]

4 Orderings of Fuzzy Sets

The ordering-based modifiers defined above can serve as the basis for defining an ordering of fuzzy sets. Let us briefly recall the well-known partial ordering of real intervals

$$[a,b] \leq_I [c,d] \iff (a \leq c \land b \leq d)$$

The inequality $a \leq c$ means that there are no elements of the set [c, d] which are below the entire interval [a, b]. The inequality $b \leq d$, analogously, means that there are no elements of [a, b] which lie completely above [c, d]. As ATL(A) gives the "left flank" of fuzzy set A and ATM(A) gives the "right flank" of fuzzy set A, the generalization of the above interval ordering is straightforward.

Definition 4. Let *L* be a fuzzy ordering on *X*. Then the relation \preceq_L on $\mathcal{F}(X)$ is defined in the following way (with the usual notation that $A \subseteq B$ if and only if $A(x) \leq B(x)$ for all $x \in X$):

$$A \preceq_L B \Leftrightarrow (\operatorname{ATL}(A) \supseteq \operatorname{ATL}(B) \land \operatorname{ATM}(A) \subseteq \operatorname{ATM}(B))$$

It is trivial to see that \leq_L is a reflexive and transitive fuzzy relation on $\mathcal{F}(X)$. Its non-antisymmetry is characterized in the following way.

Theorem 1. The following holds for all fuzzy subsets $A, B \in \mathcal{F}(X)$, where $ECX(A) = ATL(A) \cap ATM(A)$ (with \cap being the intersection with respect to the minimum t-norm):

$$(A \preceq_L B \land A \succeq_L B) \Leftrightarrow \operatorname{ECX}(A) = \operatorname{ECX}(B)$$

For more detailed argumentation and interpretation of this correspondence, see [9,11]. We just mention that there is a very close connection between the operator ECX and the convex hull. Moreover, it is worth to note that this approach—if the crisp linear ordering on the real numbers is considered—coincides with well-known fuzzification of the linear ordering of real numbers based on Zadeh's extension principle [16–18].

The reader should be aware that, in any case, different heights of two fuzzy sets immediately imply incomparability with respect to \leq_L . Therefore, we restrict our considerations to normal fuzzy sets in the following (i.e. fuzzy sets A for which at least one element exists that fulfills A(x) = 1).

5 Fuzzy Orderings of Fuzzy Sets

If we consider the two convex fuzzy quantities A_3 and B_3 shown in Figure 3, it is easy to see that, if we construct \leq_L by means of choosing L to be the natural ordering \leq of real numbers, these two triangular fuzzy quantities are incomparable. Even if a fuzzy ordering L fuzzifying \leq is taken, the situation cannot be better. What we are facing here is exactly the inappropriateness of comparing vague phenomena crisply, which leads to artificial preciseness. In this section, we want to overcome this problem by allowing intermediate degrees to which a fuzzy set is smaller or equal than another. For this purpose, let us reconsider the definition of $A \leq_L B$:

$$\operatorname{ATL}(A) \supseteq \operatorname{ATL}(B) \wedge \operatorname{ATM}(A) \subseteq \operatorname{ATM}(B)$$
 (1)

If we want to make this crisp expression fuzzy, we have to specify (1) a fuzzy concept of subsethood and (2) a conjunction operation. Now assume that we are given a fuzzy relation SIM on $\mathcal{F}(X)$ measuring the similarity of fuzzy sets (commonly called *similarity measure*) and that we are given a fuzzy relation INCL on $\mathcal{F}(X)$ that measures the degree to which a fuzzy set is a subset of another. Then we can directly write down a generalized definition of (1) (with \tilde{T} being some t-norm):

$$\mathcal{L}_L(A, B) = \tilde{T}(\mathrm{INCL}(\mathrm{ATL}(B), \mathrm{ATL}(A)), \mathrm{INCL}(\mathrm{ATM}(A), \mathrm{ATM}(B)))$$

The question arises which properties we can expect from the fuzzy relation \mathcal{L}_L or, in other words, which requirements need to be fulfilled in order to achieve reasonable properties of \mathcal{L}_L . The following theorem tries to give an answer.

Theorem 2. If SIM is a fuzzy equivalence relation on $\mathcal{F}(X)$ with respect to some t-norm T' and INCL is a T'-SIM-ordering on $\mathcal{F}(X)$, then \mathcal{L}_L as defined above with $\tilde{T} = \min$ is a fuzzy ordering with respect to T' and the T'-equivalence

$$\mathcal{E}_L(A, B) = \operatorname{SIM}(\operatorname{ECX}(A), \operatorname{ECX}(B)).$$

Note that the t-norm T is the one that the fuzzy relations L and E are taking into account and the t-norm T' is the one that relates to INCL and SIM. These two t-norms need not be equal, therefore, we explicitly distinguish between these two here.

It remains to clarify how SIM and INCL can be chosen such that the conditions of Theorem 2 are fulfilled. We will now provide an approach that is based on residual implications.

Definition 5. For a left-continuous t-norm T', the *residual implication* (residuum) is defined as

$$\vec{T}'(x,y) = \sup\{z \in [0,1] \mid T'(x,z) \le y\},\$$

while the corresponding *biimplication* (equivalence) is defined as

$$\vec{T}'(x,y) = \min\left(\vec{T}'(x,y), \vec{T}'(y,x)\right)$$

In the framework of many-valued predicate logics based on residuated lattices [19, 20], it is natural to define the degree of inclusion of a fuzzy set A in another fuzzy set B as [21, 15]

$$\operatorname{INCL}_{T'}(A, B) = \inf_{x \in X} \vec{T}' \big(A(x), B(x) \big).$$

Theorem 3. The relation $\text{INCL}_{T'}$ is a fuzzy ordering on $\mathcal{F}(X)$ with respect to T' and the fuzzy equivalence relation

$$\operatorname{SIM}_{T'}(A,B) = \inf_{x \in X} \vec{T}'(A(x), B(x)).$$

As a consequence, we obtain that the fuzzy relation

$$\mathcal{L}_{L,T'}(A,B) = \min \left(\text{INCL}_{T'}(\text{ATL}(B), \text{ATL}(A)), \\ \text{INCL}_{T'}(\text{ATM}(A), \text{ATM}(B)) \right)$$

is fuzzy ordering on $\mathcal{F}(X)$ with respect to T' and the fuzzy equivalence relation

$$\mathcal{E}_{L,T'}(A,B) = \operatorname{SIM}_{T'}(\operatorname{ECX}(A), \operatorname{ECX}(B))$$

So far, it remains an open question in which way the crisp ordering \lesssim_L and the fuzzy ordering $\mathcal{L}_{L,T'}$ are related to each other. The next result gives an exhaustive answer:

Theorem 4. The following characterization of the kernel of $\mathcal{L}_{L,T'}$ holds (for all fuzzy sets $A, B \in \mathcal{F}(X)$):

$$\mathcal{L}_{L,T'}(A,B) = 1 \iff A \preceq_L B$$

The relationship between $\mathcal{E}_{L,T'}$ and the symmetric kernel of \leq_L is given analogously (for all fuzzy sets $A, B \in \mathcal{F}(X)$):

$$\mathcal{E}_{L,T'}(A,B) = 1 \iff \operatorname{ECX}(A) = \operatorname{ECX}(B)$$

In particular, this entails that \leq_L is a subrelation of $\mathcal{L}_{L,T'}$ which implies that the comparability of two fuzzy sets with respect to \mathcal{L}_L cannot be worse than comparability with respect to \leq_L .

Example 1. Let us choose L to be the crisp linear ordering of real numbers (i.e. $L = \chi_{\leq}$) and $T'(x, y) = T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0)$. For the fuzzy quantities shown in Figures 1 and 2, we obtain the following:

$\mathcal{L}_{L,T'}(A_1, B_1) = 1$	$\mathcal{L}_{L,T'}(A_2, B_2) = 1$
$\mathcal{L}_{L,T'}(B_1,A_1) = 0$	$\mathcal{L}_{L,T'}(B_2, A_2) = 0$
$\mathcal{E}_{L,T'}(B_1, A_1) = 0$	$\mathcal{E}_{L,T'}(B_2, A_2) = 0$

For the examples shown in Figures 3 and 4, the following encouraging results are obtained:

$$\mathcal{L}_{L,T'}(A_3, B_3) = 0.9 \qquad \qquad \mathcal{L}_{L,T'}(A_4, B_4) = \frac{5}{8} \\ \mathcal{L}_{L,T'}(B_3, A_3) = 0 \qquad \qquad \mathcal{L}_{L,T'}(B_4, A_4) = \frac{5}{12} \\ \mathcal{E}_{L,T'}(B_3, A_3) = 0 \qquad \qquad \mathcal{E}_{L,T'}(B_4, A_4) = \frac{5}{12} \\ \mathcal{E}_{L,T'}(B_4, A_4) = \frac{5}$$

6 Concluding Remarks

In this paper, a general fuzzy concept for ordering fuzzy sets with respect to fuzzy orderings was introduced. We have seen that this concept smoothly fits into the framework of fuzzy orderings. The examples and studies in this paper were based on implication-based inclusion and similarity measures (INCL_{T'} and $\text{SIM}_{T'}$), which is a relatively restrictive setting. The future challenge is now to find and study other inclusion and similarity measures that provide reasonable properties in the spirit of Theorem 2, but allow more freedom.

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Fig. 4. A_4 (solid), B_4 (dashed)