Applications of Fuzzy Orderings: An Overview

Ulrich Bodenhofer

Software Competence Center Hagenberg
A-4232 Hagenberg, Austria
e-mail: ulrich.bodenhofer@scch.at

Abstract. This contribution aims at the dissemination of a similarity-based generalization of fuzzy orderings, however, not from a purely theoretical perspective, but from the viewpoint of possible practical applications. After a short motivation of similarity-based fuzzy orderings, we consider four case studies with the aim to illustrate the application potential of fuzzy orderings—flexible query answering systems, ordering-based modifiers, orderings of fuzzy sets, and interpretability of linguistic variables.

1 Introduction

There are two most fundamental relational concepts in mathematics which accompany mathematicians as well as computer scientists and engineers throughout their life in science—*equivalence relations* (reflexive, symmetric, and transitive relations) and *(partial) orderings* (reflexive, antisymmetric, and transitive relations).

It is not surprising that, within the early gold rush of fuzzification of virtually any classical mathematical concept, these two fundamental types of relations did not have to await the introduction of their fuzzy counterparts for a long time [44].

Fuzzy equivalence relations are now well-accepted concepts for expressing equivalence/equality in vague environments [26, 30, 39, 40] (in contrast to Zadeh's original definition, now with the additional degree of freedom that the conjunction in transitivity may be modeled by an arbitrary triangular norm, i.e. a binary operation on the unit interval that is commutative, associative, non-decreasing, and has 1 as neutral element [28]).

Definition 1. A binary fuzzy relation E on a domain X is called *fuzzy equivalence relation* with respect to a t-norm T, for brevity T*-equivalence*, if and only if the following three axioms are fulfilled for all $x, y, z \in X$:

(i) Reflexivity: $E(x, x) = 1$
(ii) Symmetry: $E(x, y) = E(y, x)$
(iii) T-transitivity: $T\big(E(x, y), E(y, z)\big) \leq E(x, z)$

In the meantime, fuzzy equivalence relations have turned out to be helpful tools in various disciplines, in particular, as soon as the interpretation of fuzzy sets, partitions, and controllers [22,27,30,40] is concerned. More direct practical applications have emerged in flexible query answering systems [24, 33] and fuzzy databases in general [35].

Fuzzy (partial) orderings have been introduced more or less in parallel with fuzzy equivalence relations [44], however, they have never played a significant role in real-world applications.

Definition 2. A binary fuzzy relation R on a domain X is called *fuzzy (partial) ordering* with respect to a t-norm T, for brevity T*-ordering*, if and only if the following three axioms hold (for all $x, y, z \in X$):

(i) Reflexivity: $R(x,x) = 1$
(ii) T-antisymmetry: $T\big(R(x,y), R(y,x)\big) = 0$ whenever $x \neq y$
(iii) T-transitivity: $T\big(R(x,y), R(y,z)\big) \leq R(x,z)$

This kind of fuzzy relations has only been of some importance in the preference modeling niche [19,34], but so far not at all in fuzzy systems applications, although almost every fuzzy system inherently uses ordinal structures in vague environments—there might be only a very small minority of fuzzy systems in which expressions like '*small*', '*medium*', or '*large*' do not occur.

This paper advocates a "similarity-based" generalization of fuzzy orderings, however, not from the pure mathematical viewpoint of logic or algebra (for what we would like to refer to the extensive studies in [3,4,23]). Instead, we attempt to demonstrate the potential for applications by means of considering comprehensive overviews of four case studies. Those are flexible query answering systems, ordering-based modifiers, and orderings of fuzzy sets. Finally, we also discuss the interpretability property, for which orderings of fuzzy sets are of fundamental importance.

2 "Similarity-Based" Fuzzy Orderings

In the crisp case, equivalence relations and orderings are both special cases of preorderings (reflexive and transitive relations). While equivalence relations can be constructed as symmetric kernels of preorderings, orderings are obtained from preorderings by factorization with respect to the symmetric kernel. In the fuzzy case, the first correspondence still holds, i.e. fuzzy equivalence relations are uniquely described as symmetric kernels of fuzzy preorderings (reflexive and T-transitive fuzzy relations) [40]. Fuzzy orderings, however, have not been constructed as factorizations of fuzzy preorderings. Instead, Definition 2 is just a straightforward generalization of the three classical axioms of orderings, but without taking the deeper algebraic background into account. This important fact has been addressed first in [23], where a generalization is proposed which complies with these deep correspondences by taking the strong relationship between ordering and equivalence ("similarity") into account. This idea has been revitalized and further developed in [3, 4]. In these two papers, it is also shown that several serious practical shortcomings of fuzzy orderings in the sense of Definition 2 are resolved in the more general framework.

Definition 3. A fuzzy relation $L : X^2 \to [0,1]$ is called *fuzzy ordering* with respect to a t-norm T and a T-equivalence E, for brevity *T-E-ordering*, if and only if it fulfills the following three axioms for all $x, y \in X$:

(i) E-Reflexivity: $E(x,y) \leq L(x,y)$
(ii) T-E-antisymmetry: $T\big(L(x,y), L(y,x)\big) \leq E(x,y)$
(iii) T-transitivity: $T\big(L(x,y), L(y,z)\big) \leq L(x,z)$

Note that the above formulations of reflexivity and antisymmetry (both with respect to an underlying fuzzy equivalence relation) emulate the classical factorization construction in an elegant way without using factor sets explicitly. Since it is not the major concern of this paper, we will not repeat all representation and construction results and refer to [3,4] instead. However, we will mention an important subclass which will be of particular practical interest throughout the remaining paper.

Definition 4. A T-E-ordering L is called *strongly linear* if and only if, for all $x, y \in X$,

$$\max\big(L(x,y), L(y,x)\big) = 1.$$

As the term "strong linearity" suggests, this property can be considered as a generalization of the classical linearity property. However, strong linearity is usually too strong a requirement to be acceptable as a general concept of linearity in specific logical and algebraic terms [8]. Anyway, as we will see next, strongly linear fuzzy orderings have a certain practical importance.

Definition 5. A crisp ordering $\preceq$ on a domain X and a T-equivalence $E : X^2 \to [0,1]$ are called *compatible*, if and only if the following holds for all $x, y, z \in X$:

$$x \preceq y \preceq z \Rightarrow E(x,z) \leq \min\big(E(x,y), E(y,z)\big)$$

Compatibility between a crisp ordering $\preceq$ and a fuzzy equivalence relation E can be interpreted as follows: the two outer elements of an ordered three-element chain are at most as similar as any two inner elements.

Theorem 1. [3, 4] *Consider a fuzzy relation L on a domain X and a T-equivalence E. Then the following two statements are equivalent:*

(i) L is a strongly linear T-E-ordering.
(ii) There exists a linear ordering $\preceq$ the relation E is compatible with such that L can be represented as follows:

$$L(x,y) = \begin{cases} 1 & \text{if } x \preceq y \\ E(x,y) & \text{otherwise} \end{cases} \tag{1}$$

Theorem 1 allows to consider strongly linear T-E-orderings as "linear orderings with imprecision", which are common phenomena in everyday life. Consider, for instance, the old example of comparing the heights of people. Although we have a clear crisp concept for ordering heights (which are just positive real numbers), there is undoubtedly a certain tolerance for imprecision or indistinguishability in the way we actually perform such a comparison. Similar situations occur in virtually any application where values have to be processed for which a crisp ordering would exist, but where small differences are either impossible to grasp (e.g. due to limited measurement accuracy) or simply not even necessary to be considered.

3 Flexible Query Answering Systems

Flexible query answering systems can be considered as a subbranch of fuzzy database systems [35]. They address the problem how query interfaces to conventional databases with crisp data can be extended such that a flexible interpretation of queries is possible [11, 12, 17, 24, 25, 31, 33, 37, 38]—with the motivation to suggest alternatives close to the query, in particular, in cases where no record matches the query in an exact way. Existing flexible query answering systems use fuzzy equivalence relations or other concepts of gradual similarity in order to achieve this goal.

While most other systems use similarity tables to compute the degrees of fulfillment of a query, the developers of the so-called *Vague Query System (VQS)* [31,32] have chosen the pragmatic way to map non-numeric to numeric values (e.g. color labels to RGB values) in order to apply simple Euclidean distances to compute degrees of similarity between a database records and queries.

Almost all flexible query answering systems are only able to deal with degrees of similarity, although ordering-based queries would also be highly important from a practical point of view. For instance, a user sending a query to a tourist information system may not primarily be interested in hotel rooms that cost \$70 per night, but rather in those which do not exceed this limit. In either case, a result of \$70.50 should still be in the result set,[1] since 50 cents do not make a significant difference. It is easy to observe that this is exactly a situation as described at the end of the previous section—we have a clear concept of crisp (even linear) ordering, but with a certain tolerance for indistinguishability. There are attempts in this direction [25], however, they are not based on the proposed framework of similarity-based fuzzy orderings which would be able to provide a simple and sound basis.

Therefore, under the assumption that we have a linear ordering for some attribute and a concept of similarity modeled by a fuzzy equivalence relation which is compatible with the ordering in the sense of Definition 5, Theorem 1

[1] Flexible query answering systems usually give sorted lists of results ranked according to the similarity between the records and the query.

gives a clear hint how the semantics of ordering-based queries can be defined. In [9, 10], an extension of the existing VQS system is presented which makes use of fuzzy orderings. In the new variant, VQS uses an extension of SQL in which the conditions "IS", "IS AT LEAST", "IS AT MOST", and "IS WITHIN" can be interpreted in a fuzzy way (generalizing the standard SQL constructs "=", ">=", '<=", and "BETWEEN", respectively). Provided that we are given an attribute on a domain X for which we know a crisp linear ordering $\preceq$ and a T-equivalence E which is compatible to $\preceq$, the fuzzy relation defined in (1) is a strongly linear T-E-ordering. Then we can compute the degrees of fulfillment of the following query fragments in the following way (for a query value q and a record x):

$$\begin{aligned}
t(\text{“}x \text{ IS } q\text{”}) &= E(q,x)\\
t(\text{“}x \text{ IS AT LEAST } q\text{”}) &= L(q,x)\\
t(\text{“}x \text{ IS AT MOST } q\text{”}) &= L(x,q)\\
t(\text{“}x \text{ IS WITHIN } (a,b)\text{”}) &= T\big(L(\min(a,b),x), L(x,\max(a,b))\big)
\end{aligned}$$

Note that the definition of T-equivalences from Euclidean distances is straightforward by the duality of pseudo-metrics and T-equivalences (with the additional requirement that T has to be continuous and Archimedean) [9, 10, 15, 16, 28]. Therefore, this extension smoothly integrates into the existing VQS framework. For extensive details, see [9].

4 Ordering-Based Modifiers

Already in their beginning, fuzzy systems were considered as appropriate tools for controlling complex systems and for carrying out complicated decision processes [45]. It is well-known and easy to see that, if rule bases are represented as complete tables, the number of rules grows exponentially with the number of variables—a fact which has to be regarded as a severe practical limitation. Beside others, the integration of advanced linguistic constructs such as modifiers (adverbs) may be considered as one possible measure to keep rule bases compact. Ordering-based modifiers, such as *'at least'*, *'at most'*, *'between'*, for instance, could be used for grouping neighboring rules with the same consequents, thereby, reducing the number of rules while improving expressiveness and interpretability.

The most elegant and efficient way to define the semantics of such modifiers is to have an unambiguous computational model to construct, for example, a fuzzy set with the meaning "at least A" for any given fuzzy set A. Images with respect to fuzzy orderings provide such a methodology, even with the freedom to take an underlying context of indistinguishability into account [2, 7].

Definition 6. [20] Let R be a binary fuzzy relation on a domain X and let T be a triangular norm. For a given fuzzy set A on X, the image of A with respect to R is defined as

$$R{\uparrow}A(x) = \sup\{T\big(A(y), R(y,x)\big) \mid y \in X\}.$$

In case that the t-norm T is left-continuous [28] and if R is a reflexive and T-transitive (i.e. a so-called T-preordering), then the operator $R{\uparrow}$ fulfills certain extremal properties and is idempotent [3]. For a T-equivalence E and a fuzzy set A, $E{\uparrow}A$ is often called *extensional hull* (of A) [26,30], which we will denote with the symbol $\mathrm{EXT}(A)$ in the following.

Most importantly, if L is a T-E-ordering, $L{\uparrow}A$ is nothing else than the fuzzy set which contains A and "all those elements that are above the elements of A" [3,2]. Therefore, $L{\uparrow}A$ models the expression *'at least A'*) (with respect to a given fuzzy ordering L). We will abbreviate $L{\uparrow}A$ with $\mathrm{ATL}(A)$ in the following. Analogously, for the image with respect to the inverse fuzzy ordering $G(x,y) = L(y,x)$, the symbol $\mathrm{ATM}(A) = G{\uparrow}A$—standing for *'at most A'*—will be used.

If we denote the image operator of a crisp ordering $\preceq$ with LTR and the image operator of its inverse with RTL, then the following simplified representations hold [3]:

$$\begin{aligned}\mathrm{LTR}(A)(x) &= \sup\{A(y) \mid y \preceq x\}\\ \mathrm{RTL}(A)(x) &= \sup\{A(y) \mid x \preceq y\}\end{aligned}$$

Furthermore, let us make the following definitions:

$$\begin{aligned}\mathrm{CVX}(A)(x) &= \min\big(\mathrm{LTR}(A)(x), \mathrm{RTL}(A)(x)\big)\\ \mathrm{ECX}(A)(x) &= \min\big(\mathrm{ATL}(A)(x), \mathrm{ATM}(A)(x)\big)\end{aligned}$$

It is relatively easy to see that the operator CVX yields the smallest convex fuzzy superset of A if we consider a fuzzy set B as convex if and only if the following holds (for all $x, y, z \in X$) [3]:

$$x \preceq y \preceq z \;\Rightarrow\; B(y) \geq \min\big(B(x), B(z)\big)$$

The following theorem gives a clear answer how the operators ATL, ATM, and ECX are represented if a T-E-ordering is strongly linear—a case that is, as already mentioned, of vital practical interest.

Theorem 2. [3] *Provided that L is a strongly linear T-E-ordering on a domain X such that the representation* (1) *holds for some crisp linear ordering $\preceq$, the following equalities hold for every fuzzy set A on X:*

$$\begin{aligned}\mathrm{ATL}(A) &= \mathrm{LTR}(\mathrm{EXT}(A)) = \mathrm{EXT}(\mathrm{LTR}(A))\\ \mathrm{ATM}(A) &= \mathrm{RTL}(\mathrm{EXT}(A)) = \mathrm{EXT}(\mathrm{RTL}(A))\\ \mathrm{ECX}(A) &= \mathrm{CVX}(\mathrm{EXT}(A)) = \mathrm{EXT}(\mathrm{CVX}(A))\end{aligned}$$

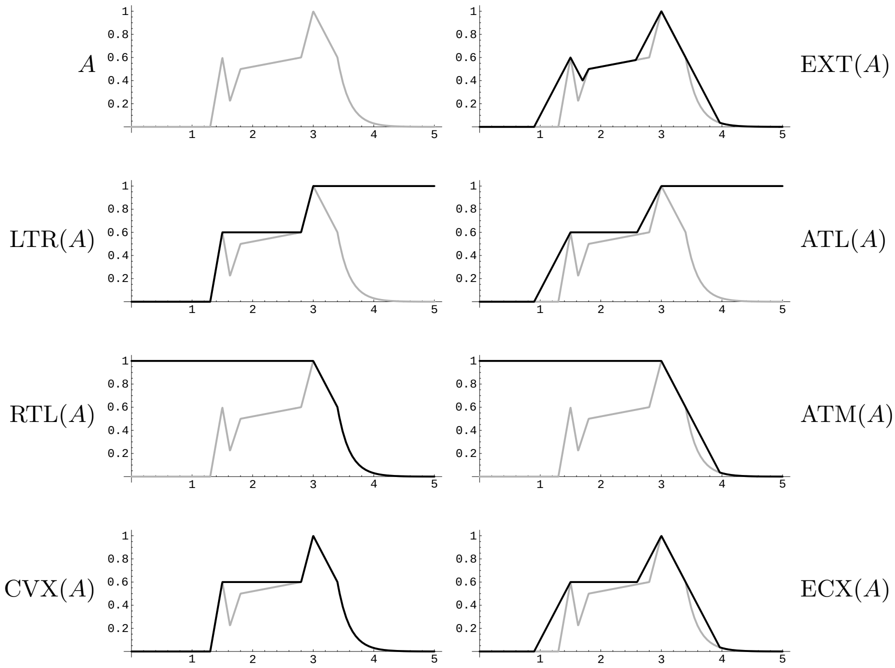

Fig. 1. A fuzzy set and the results obtained by applying various ordering-based modifiers.

In order to demonstrate the actual meaning of the operators ATL and ATM and the correspondences of Theorem 2, let us consider the following two fuzzy relations on the real numbers:

$$E(x, y) = \max(1 - |x - y|, 0)$$

$$L(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ \max(1 - x + y, 0) & \text{otherwise} \end{cases}$$

One easily verifies that E is a $T_{\mathbf{L}}$-equivalence on the real numbers and that L is a $T_{\mathbf{L}}$-E-ordering, where $T_{\mathbf{L}}$ stands for the Łukasiewicz t-norm $T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0)$. Figure 1 shows a non-trivial (non-convex and non-extensional) fuzzy set and the results obtained by the seven operators we have discussed in this section.

5 Orderings of Fuzzy Sets

Orderings/rankings of fuzzy sets play an important role in fuzzy decision analysis, but also in linguistic approximation, rule interpolation [29], and many other disciplines. Most previous approaches have in common that they are restricted to certain subclasses of fuzzy sets and that they only work for fuzzy subsets of the real numbers (see [41, 42] for detailed reviews).

Based on the definition of ordering-based modifiers (as described in Section 4), it is possible to define a preordering of arbitrary fuzzy sets on any domain for which a T-E-ordering L is known [3, 5]. This relation is defined as

$$A \preceq_L B \;\Leftrightarrow\; \big(\mathrm{ATL}(B) \subseteq \mathrm{ATL}(A) \text{ and } \mathrm{ATM}(A) \subseteq \mathrm{ATM}(B)\big),$$

where $\subseteq$ denotes the usual crisp inclusion of fuzzy sets, i.e. $A \subseteq B$ if and only if $A(x) \leq B(x)$ for any $x \in X$ [43]. It is easy to see (compare with Fig. 1) that the first inclusion $\mathrm{ATL}(B) \subseteq \mathrm{ATL}(A)$ corresponds to the fact that the left flank of A is above (to the left) of the left flank of B, while the second inclusion $\mathrm{ATM}(A) \subseteq \mathrm{ATM}(B)$ determines whether the right flank of A is below (to the left) of the right flank of B.

The next theorem provides a unique characterization of non-antisymmetry of $\preceq_L$.

Theorem 3. [3] *Consider a t-norm T and a T-E-equivalence E. If L is a T-E-ordering on the domain X, then $\preceq_L$ is reflexive, transitive, and antisymmetric up to the equivalence relation*

$$A \sim_L B \;\Leftrightarrow\; \mathrm{ECX}(A) = \mathrm{ECX}(B).$$

Corollary 1. [3] *Consider a crisp ordering $\preceq$. Then*

$$A \preceq_I B \;\Leftrightarrow\; \big(\mathrm{LTR}(A) \supseteq \mathrm{LTR}(B) \textit{ and } \mathrm{RTL}(A) \subseteq \mathrm{RTL}(B)\big)$$

is a reflexive and transitive relation which is antisymmetric up to the equivalence relation

$$A \sim_I B \;\Leftrightarrow\; \mathrm{CVX}(A) = \mathrm{CVX}(B).$$

In case that L is a strongly linear T-E-ordering, thereby admitting resolution (1), Theorems 2 and 3 together with Corollary 1 provide very specific information about the non-antisymmetry of the relation $\preceq_L$.

This approach can be applied to any kind of fuzzy sets on a domain for which a crisp or fuzzy ordering is known, with the only restriction that these ordering methods cannot distinguish between fuzzy sets with equal (extensional) convex hulls. In particular, no special assumptions concerning the structure of the space X (e.g., linearity of the ordering, restriction to real numbers or intervals, etc.) have to be made. Note that the above relations are not complete in the sense that any two fuzzy sets are comparable. Since orderings of fuzzy sets are still more general relations than interval orderings, completeness would not be a natural assumption anyway [3, 5].

6 Interpretability of Linguistic Variables

The main difference between fuzzy systems and other control or decision support systems is that they are parameterized in an interpretable way—by means of rules consisting of linguistic expressions. If a fuzzy system is constructed from expert knowledge, the resulting systems are most often interpretable in the sense that a human can easily guess what the system is approximately doing (afterwards, without even being involved in the design process). If automatic tuning or machine learning methods are applied to construct a fuzzy system according to some design goals (e.g. fitting of sample data), the result is—depending on the method used—still a fuzzy system from the formal point of view, but most often lacks this crucial property that a human is able to guess its qualitative behavior easily.

This important fact is increasingly being recognized in the fuzzy control community [1,13,14,18,36]. However, most researchers have approached this problem simply by making some common sense assumptions about the shape and mutual overlapping of membership functions, without even attempting to ask in detail what *interpretability* actually is. In a recent investigation [6], interpretability is defined as the *possibility to estimate a fuzzy system's behavior by reading and understanding the rule base only.* The framework for expressing interpretability as a mathematical property is based on the fact that humans do have a qualitative understanding of the semantics of linguistic labels, such as "small", "medium", or "large". If this information can be formulated by means of (fuzzy) relations and if canonical counterparts of these relations exist on the semantic level (i.e. the fuzzy sets modeling the linguistic expressions), interpretability can be viewed as the preservation of the relationships between the labels by the corresponding fuzzy sets.

As a simplistic example, assume that we are given a linguistic variable which may take the values "small", "medium", or "large". It is obvious that humans associate a certain ordinal structure with these three labels. The counterpart of this ordering on the semantic level is nothing else than an ordering of fuzzy sets—which brings us back to the concepts discussed in Section 5. In [6], a more elaborate (and less trivial) example is given which also makes use of fuzzy orderings and orderings of fuzzy sets.

Taking interpretability in the automatic construction of fuzzy systems into account implies that some mutual relationships between the different fuzzy sets have to be preserved. From the computational point of view, this leads to constrained optimization problems, which are always more difficult to handle. In order to tackle this problem, more sophisticated optimization algorithms are necessary (see [13] for an overview of different possible approaches, another promising approach based on numerical optimization is presented in [21]).

7 Conclusion

This paper is intended as a pleading for the importance of fuzzy orderings in applications aside of preference modeling and decision analysis. In order to support this claim, four potential fields of practical applications have been discussed—one from the fuzzy database area, the other three rather from the fuzzy systems/fuzzy control area. These four case studies clearly underline that fuzzy orderings are not just of pure theoretical interest, but can also have fruitful practical applications.

Acknowledgements

The author gratefully acknowledges support by the Austrian Government, the State of Upper Austria, and the Johannes Kepler University Linz in the framework of the K*plus* Competence Center Program.

References

1. R. Babuška. Construction of fuzzy systems—interplay between precision and transparency. In *Proc. European Symposium on Intelligent Techniques (ESIT 2000)*, pages 445–452, Aachen, September 2000.
2. U. Bodenhofer. The construction of ordering-based modifiers. In G. Brewka, R. Der, S. Gottwald, and A. Schierwagen, editors, *Fuzzy-Neuro Systems '99*, pages 55–62. Leipziger Universitätsverlag, 1999.
3. U. Bodenhofer. *A Similarity-Based Generalization of Fuzzy Orderings*, volume C 26 of *Schriftenreihe der Johannes-Kepler-Universität Linz*. Universitätsverlag Rudolf Trauner, 1999.
4. U. Bodenhofer. A similarity-based generalization of fuzzy orderings preserving the classical axioms. *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems*, 8(5):593–610, 2000.
5. U. Bodenhofer. A general framework for ordering fuzzy sets. In B. Bouchon-Meunier, J. Guitiérrez-Ríoz, L. Magdalena, and R. R. Yager, editors, *Technologies for Constructing Intelligent Systems 1: Tasks*, volume 89 of *Studies in Fuzziness and Soft Computing*, pages 213–224. Physica-Verlag, Heidelberg, 2002.
6. U. Bodenhofer and P. Bauer. A formal model of interpretability of linguistic variables. In J. Casillas, O. Cordón, F. Herrera, and L. Magdalena, editors, *Interpretability Issues in Fuzzy Modeling*, volume 128 of *Studies in Fuzziness and Soft Computing*, pages 524–545. Springer, Berlin, 2003.
7. U. Bodenhofer, M. De Cock, and E. E. Kerre. Openings and closures of fuzzy preorderings: Theoretical basics and applications to fuzzy rule-based systems. *Int. J. General Systems*, 32(4):343–360, 2003.
8. U. Bodenhofer and F. Klawonn. A formal study of linearity axioms for fuzzy orderings. *Fuzzy Sets and Systems*. (in press).
9. U. Bodenhofer and J. Küng. Fuzzy orderings in flexible query answering systems. *Soft Computing*. (in press).

10. U. Bodenhofer and J. Küng. Enriching vague queries by fuzzy orderings. In *Proc. 2nd Int. Conf. in Fuzzy Logic and Technology (EUSFLAT 2001)*, pages 360–364, Leicester, UK, September 2001.
11. P. Bosc, L. Duval, and O. Pivert. Value-based and representation-based querying of possibilistic databases. In G. Bordogna and G. Pasi, editors, *Recent Issues on Fuzzy Databases*, volume 53 of *Studies in Fuzziness and Soft Computing*, pages 3–27. Physica-Verlag, Heidelberg, 2000.
12. P. Bosc and O. Pivert. SQLf: A relational database language for fuzzy querying. *IEEE Trans. Fuzzy Systems*, 3:1–17, 1995.
13. J. Casillas, O. Cordón, F. Herrera, and L. Magdalena, editors. *Interpretability Issues in Fuzzy Modeling*, volume 128 of *Studies in Fuzziness and Soft Computing*. Springer, Berlin, 2003.
14. O. Cordón and F. Herrera. A proposal for improving the accuracy of linguistic modeling. *IEEE Trans. Fuzzy Systems*, 8(3):335–344, June 2000.
15. B. De Baets and R. Mesiar. Pseudo-metrics and T-equivalences. *J. Fuzzy Math.*, 5(2):471–481, 1997.
16. B. De Baets and R. Mesiar. Metrics and T-equalities. *J. Math. Anal. Appl.*, 267:331–347, 2002.
17. D. Dubois and H. Prade. Using fuzzy sets in flexible querying: Why and how? In *Proc. Workshop on Flexible Query-Answer Systems (FQAS'96)*, pages 89–103, Roskilde, May 1996.
18. J. Espinosa and J. Vandewalle. Constructing fuzzy models with linguistic integrity from numerical data—AFRELI algorithm. *IEEE Trans. Fuzzy Systems*, 8(5):591–600, October 2000.
19. J. Fodor and M. Roubens. *Fuzzy Preference Modelling and Multicriteria Decision Support*. Kluwer Academic Publishers, Dordrecht, 1994.
20. S. Gottwald. *Fuzzy Sets and Fuzzy Logic*. Vieweg, Braunschweig, 1993.
21. J. Haslinger, U. Bodenhofer, and M. Burger. Data-driven construction of Sugeno controllers: Analytical aspects and new numerical methods. In *Proc. Joint 9th IFSA World Congress and 20th NAFIPS Int. Conf.*, pages 239–244, Vancouver, July 2001.
22. U. Höhle. Fuzzy equalities and indistinguishability. In *Proc. EUFIT'93*, volume 1, pages 358–363, 1993.
23. U. Höhle and N. Blanchard. Partial ordering in L-underdeterminate sets. *Inform. Sci.*, 35:133–144, 1985.
24. T. Ichikawa and M. Hirakawa. ARES: A relational database with the capability of performing flexible interpretation of queries. *IEEE Trans. Software Eng.*, 12(5):624–634, 1986.
25. J. Kacprzyk and S. Zadrozny. FQUERY for Access: Fuzzy querying for a Windows-based DBMS. In P. Bosc and J. Kacprzyk, editors, *Fuzziness in Database Management Systems*, volume 5 of *Studies in Fuzziness and Soft Computing*, pages 415–433. Physica-Verlag, Heidelberg, 1995.
26. F. Klawonn. Fuzzy sets and vague environments. *Fuzzy Sets and Systems*, 66:207–221, 1994.
27. F. Klawonn and J. L. Castro. Similarity in fuzzy reasoning. *Mathware Soft Comput.*, 3(2):197–228, 1995.
28. E. P. Klement, R. Mesiar, and E. Pap. *Triangular Norms*, volume 8 of *Trends in Logic*. Kluwer Academic Publishers, Dordrecht, 2000.
29. L. T. Kóczy and K. Hirota. Ordering, distance and closeness of fuzzy sets. *Fuzzy Sets and Systems*, 59(3):281–293, 1993.

30. R. Kruse, J. Gebhardt, and F. Klawonn. *Foundations of Fuzzy Systems*. John Wiley & Sons, New York, 1994.
31. J. Küng and J. Palkoska. VQS—a vague query system prototype. In *Proc. 8th Int. Workshop on Database and Expert Systems Applications (DEXA '97)*, pages 614–618. IEEE Computer Society Press, Los Alamitos, CA, 1997.
32. J. Küng and J. Palkoska. Vague joins—an extension of the vague query system VQS. In *Proc. 9th Int. Workshop on Database and Expert Systems Applications (DEXA '98)*, pages 997–1001. IEEE Computer Society Press, Los Alamitos, CA, 1998.
33. A. Motro. VAGUE: A user interface to relational databases that permits vague queries. *ACM Trans. Off. Inf. Syst.*, 6(3):187–214, 1988.
34. S. V. Ovchinnikov. Similarity relations, fuzzy partitions, and fuzzy orderings. *Fuzzy Sets and Systems*, 40(1):107–126, 1991.
35. F. E. Petry and P. Bosc. *Fuzzy Databases: Principles and Applications*. International Series in Intelligent Technologies. Kluwer Academic Publishers, Boston, 1996.
36. M. Setnes, R. Babuška, and H. B. Verbruggen. Rule-based modeling: Precision and transparency. *IEEE Trans. Syst. Man Cybern. C*, 28:165–169, 1998.
37. V. Tahani. A conceptual framework for fuzzy query processing: A step towards very intelligent database systems. *Inf. Proc. and Manag.*, 13:289–303, 1977.
38. Y. Takahashi. Fuzzy database query languages and their relational completeness theorem. *IEEE Trans. Knowl. Data Eng.*, 5(1):122–125, 1993.
39. E. Trillas and L. Valverde. An inquiry into indistinguishability operators. In H. J. Skala, S. Termini, and E. Trillas, editors, *Aspects of Vagueness*, pages 231–256. Reidel, Dordrecht, 1984.
40. L. Valverde. On the structure of F-indistinguishability operators. *Fuzzy Sets and Systems*, 17(3):313–328, 1985.
41. X. Wang and E. E. Kerre. Reasonable properties for the ordering of fuzzy quantities (I). *Fuzzy Sets and Systems*, 118:375–385, 2001.
42. X. Wang and E. E. Kerre. Reasonable properties for the ordering of fuzzy quantities (II). *Fuzzy Sets and Systems*, 118:387–405, 2001.
43. L. A. Zadeh. Fuzzy sets. *Inf. Control*, 8:338–353, 1965.
44. L. A. Zadeh. Similarity relations and fuzzy orderings. *Inform. Sci.*, 3:177–200, 1971.
45. L. A. Zadeh. Outline of a new approach to the analysis of complex systems and decision processes. *IEEE Trans. Syst. Man Cybern.*, 3(1):28–44, 1973.