Lexicographic Composition of Fuzzy Orderings

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Abstract

The present paper introduces an approach to construct lexicographic compositions of similarity-based fuzzy orderings. This construction is demonstrated by means of non-trivial examples. As this is a crucial feature of lexicographic composition, the preservation of linearity is studied in detail. We obtain once again that it is essential for meaningful results that the t-norm under consideration induces an involutive (strong) negation.

Keywords: fuzzy equivalence relations, fuzzy orderings, lexicographic composition, strict fuzzy orderings.

1 Introduction

Lexicographic composition is a fundamental construction principle for ordering relations. The most important feature of this construction is that the composition of two linear orderings again yields a linear ordering. Given two orderings $\leq_1$ and $\leq_2$ on non-empty domains X_1 and X_2, respectively, the lexicographic composition is an ordering $\leq'$ on the Cartesian product $X_1 \times X_2$, where $(x_1, x_2) \leq' (y_1, y_2)$ if and only if

$$(x_1 \neq y_1 \wedge x_1 \leq_1 y_1) \vee (x_1 = y_1 \wedge x_2 \leq_2 y_2). \quad (1)$$

Rewriting $x_1 \neq y_1 \wedge x_1 \leq_1 y_1$ as $x_1 <_1 y_1$ (i.e. the strict ordering induced by $\leq_1$) and taking into account that $x_1 = y_1 \vee x_1 \neq y_1$ is a tautology and that $\leq_1$ is reflexive, we obtain that (1) is equivalent to

$$(x_1 \leq_1 y_1 \wedge x_2 \leq_2 y_2) \vee x_1 <_1 y_1. \quad (2)$$

The study of fuzzy orderings can be traced back to the early days of fuzzy set theory [12, 17, 18, 22]. Partial fuzzy orderings in the sense of Zadeh [22], however, have severe shortcomings that were finally resolved by replacing the crisp equality by a fuzzy equivalence relation, thereby maintaining the well-known classical fact that orderings are obtained from preorderings by factorization [7, 2, 3, 11, 14].

In [1, 3], several methods for constructing fuzzy orderings are presented, including Cartesian products. How to transfer lexicographic composition to the fuzzy framework, however, remained an open problem. The reason why this remained an open issue for a relatively long time is that there was no meaningful concept of strict fuzzy ordering in the similarity-based framework so far. As this issue is solved by [4] now, we are able to give a solution in this paper. Detailed proofs are omitted, as they are long and technical. Details are available from the author upon request (and the reader is also referred to upcoming publications).

2 Preliminaries

For simplicity, we consider the unit interval $[0, 1]$ as our domain of truth values in this paper. Note that most results, with only minor and obvious modifications, also hold for more general structures [11, 13–15]. The symbols T, $\tilde{T}$, etc., denote left-continuous t-norms [16]. Correspondingly, $\vec{T}$ denotes the unique residual implication of T. Furthermore, we denote the residual negation of T with $N_T(x) = \vec{T}(x, 0)$.

Definition 1. A binary fuzzy relation $E : X^2 \to [0, 1]$ is called *fuzzy equivalence relation*[1] with respect to T, for brevity *T-equivalence*, if the following three axioms are fulfilled for all $x, y, z \in X$:

1. Reflexivity: $E(x, x) = 1$
2. Symmetry: $E(x, y) = E(y, x)$
3. T-transitivity:

$$T\big(E(x, y), E(y, z)\big) \leq E(x, z)$$

Definition 2. A binary fuzzy relation $L : X^2 \to [0, 1]$ is called *fuzzy ordering* with respect to T and a T-equivalence $E : X^2 \to [0, 1]$, for brevity *T-E-ordering*, if it fulfills the following three axioms for all $x, y, z \in X$:

1. E-reflexivity: $E(x, y) \leq L(x, y)$
2. T-E-antisymmetry:

$$T\big(L(x, y), L(y, x)\big) \leq E(x, y)$$

3. T-transitivity:

$$T\big(L(x, y), L(y, z)\big) \leq L(x, z)$$

Definition 3. A fuzzy relation $R : X^2 \to [0, 1]$ is called *strongly complete* if $\max(L(x, y), L(y, x)) = 1$ for all $x, y \in X$ [5,12,17]. R is called T-linear if $N_T(L(x, y)) \leq L(y, x)$ for all $x, y \in X$ [5, 14].

Definition 4. A binary fuzzy relation $S : X^2 \to [0, 1]$ is called *strict fuzzy ordering* with respect to T and a T-equivalence $E : X^2 \to [0, 1]$, for brevity *strict T-E-ordering*, if it fulfills the following axioms for all $x, x', y, y', z \in X$:

1. Irreflexivity: $S(x, x) = 0$
2. T-transitivity:

$$T\big(S(x, y), S(y, z)\big) \leq S(x, z)$$

3. E-extensionality:

$$T(E(x, x'), E(y, y'), S(x, y)) \leq S(x', y')$$

As already mentioned above, it is of vital importance for lexicographic composition how to "strictify" a given fuzzy ordering. The following theorem summarizes the most important facts.

Theorem 5. [4] *Consider a T-equivalence $E : X^2 \to [0, 1]$ and a T-E-ordering $L : X^2 \to [0, 1]$. Then the following fuzzy relation is a strict T-E-ordering on X:*

$$S(x, y) = \min(L(x, y), N_T(L(y, x)))$$

If T does not have zero divisors, the equality $S(x, y) = \min(L(x, y), N_T(E(y, x)))$ holds. Moreover, S is monotonic w.r.t. L in the following sense (for all $x, y, z \in X$).

$$T(L(x, y), S(y, z)) \leq S(x, z)$$
$$T(S(x, y), L(y, z)) \leq S(x, z)$$

S is the largest strict T-E-ordering contained in L that fulfills this kind of monotonicity.

For intersecting T-transitive fuzzy relations, the concept of domination between t-norms is of vital importance [9, 16, 19].

Definition 6. A t-norm T_1 is said to *dominate* another t-norm T_2 if, for every quadruple $(x, y, u, v) \in [0, 1]^4$, the following holds:

$$T_1\big(T_2(x, y), T_2(u, v)\big) \geq T_2\big(T_1(x, u), T_1(y, v)\big)$$

Lemma 7. [9] *Consider two t-norms T_1 and T_2. The T_2-intersection of any two arbitrary T_1-transitive fuzzy relations is T_1-transitive if and only if T_2 dominates T_1.*

3 Starting the Easy Way: One Crisp and One Fuzzy Ordering

Let us first consider the case where the primary ordering is crisp and the secondary ordering is fuzzy. As the strict ordering is only needed for the primary ordering, we do not need to take any strict fuzzy ordering into account.

[1] Note that various diverging names for this class of fuzzy relations appear in literature, like similarity relations, indistinguishability operators, equality relations, and several more [6, 15, 20, 22]

Proposition 8. *Let us consider a crisp ordering $L_1 : X_1^2 \to \{0,1\}$ and a T-E_2-ordering $L_2 : X_2^2 \to [0,1]$ (with $E_2 : X_2^2 \to [0,1]$ being a T-equivalence). Then the fuzzy relation $L : (X_1 \times X_2)^2 \to [0,1]$ defined as*

$$L((x_1,x_2),(y_1,y_2)) = \begin{cases} 1 & \text{if } x_1 \neq y_1 \text{ and} \\ & L(x_1,y_1) = 1, \\ L_2(x_2,y_2) & \text{if } x_1 = y_1, \\ 0 & \text{otherwise,} \end{cases}$$

is a fuzzy ordering w.r.t. T and the T-equivalence $\tilde{E} : (X_1 \times X_2)^2 \to [0,1]$ defined as

$$E((x_1,x_2),(y_1,y_2)) = \begin{cases} E_2(x_2,y_2) & \text{if } x_1 = y_1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that, if both components L_1 and L_2 are crisp orderings, then L as defined above is equivalent to the constructions (1) and (2).

Example 9. Consider $X_1 = X_2 = [0,4]$, let L_1 be the classical linear ordering of real numbers, and assume that L_2 is defined as follows:

$$L_2(x,y) = \max(\min(1 - x + y, 1), 0)$$

It is well-known that L_2 is a fuzzy ordering with respect to the Łukasiewicz t-norm $T_{\mathbf{L}}(x,y) = \max(x+y-1,0)$ and the $T_{\mathbf{L}}$-equivalence $E_2(x,y) = \max(1 - |x-y|, 0)$. Figure 1 shows a cut view of the fuzzy ordering L that is obtained when applying the construction from Proposition 8. The cut view has been obtained by plotting the value $L((2,2),(y_1,y_2))$ as a two-dimensional function of y_1 and y_2.

The following proposition clarifies in which way linearity of the two component orderings L_1 and L_2 is preserved by the construction in the previous proposition.

Proposition 10. *Let us make the same assumptions as in Proposition 8. If L_1 is a crisp linear ordering and L_2 is strongly complete, then L is also strongly complete. If L_1 is a crisp linear ordering and L_2 is T-linear, then L is also T-linear.*

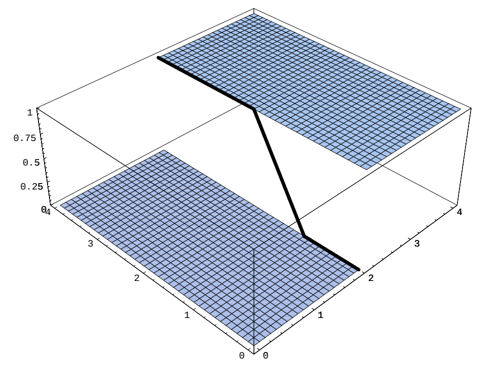

Figure 1: Cut view of a lexicographic composition of a crisp linear ordering and a fuzzy ordering according to Proposition 8

4 Lexicographic Composition of Two Non-Trivial Fuzzy Orderings

The results of the previous section have been known to the author since 1998, but they were not published so far, as they cannot be considered a full-fledged solution of the problem. So let us now consider the general case, where both components are fuzzy orderings without any further assumptions so far. The following theorem gives a general construction inspired by the classical construction (2).

Theorem 11. *Consider two T-equivalences $E_1 : X_1^2 \to [0,1]$, $E_2 : X_2^2 \to [0,1]$, a T-E_1-ordering $L_1 : X_1^2 \to [0,1]$, and a T-E_2-ordering $L_2 : X_2^2 \to [0,1]$. Moreover, let $\tilde{T}$ be a t-norm that dominates T. Then the fuzzy relation $\mathrm{Lex}_{\tilde{T},T}(L_1,L_2) : (X_1 \times X_2)^2 \to [0,1]$ defined as*

$$\begin{aligned} &\mathrm{Lex}_{\tilde{T},T}(L_1,L_2)((x_1,x_2),(y_1,y_2)) = \\ &\qquad \max\big(\tilde{T}(L_1(x_1,y_1), L_2(x_2,y_2)), \\ &\qquad\quad \min(L_1(x_1,y_1), N_T(L_1(y_1,x_1)))\big) \end{aligned} \tag{3}$$

is a fuzzy ordering w.r.t. T and the T-equivalence $\mathrm{Cart}_{\tilde{T}}(E_1,E_2) : (X_1 \times X_2)^2 \to [0,1]$ defined as the Cartesian product of E_1 and E_2:

$$\begin{aligned} &\mathrm{Cart}_{\tilde{T}}(E_1,E_2)((x_1,x_2),(y_1,y_2)) = \\ &\qquad \tilde{T}(E_1(x_1,y_1), E_2(x_2,y_2)) \end{aligned}$$

Note that, if L_1 is a crisp ordering, then $\mathrm{Lex}_{\tilde{T},T}(L_1,L_2)$ defined as in Theorem 11 coincides with the fuzzy relation L defined in Proposition 8. Consequently, if both components L_1 and L_2 are crisp orderings, then $L_{\tilde{T}}$ is equivalent to the constructions (1) and (2).

Construction (3) is based on one specific formulation of lexicographic composition, namely (2). This is just one possible way of defining lexicographic composition. It is unknown whether there are other meaningful ways to define lexicographic composition on the basis of a different formulation that is equivalent to (2) in the classical Boolean case.

Example 12. Consider again the domain $X = [0, 4]$ and consider the following three fuzzy relations on X:

$$L_3(x, y) = \max(\min(1 - \tfrac{1}{2}(x - y), 1), 0)$$
$$L_4(x, y) = \min(\exp(y - x), 1)$$
$$L_5(x, y) = \min(\exp(3(y - x)), 1)$$

L_3 is a $T_{\mathbf{L}}$-E_3-ordering with $E_3(x, y) = \max(1 - \frac{1}{2}|x - y|, 0)$. L_4 is a $T_{\mathbf{P}}$-E_4-ordering[2] with $E_4(x, y) = \exp(-|x-y|)$ and, since $T_{\mathbf{L}} \leq T_{\mathbf{P}}$, a $T_{\mathbf{L}}$-E_4-ordering as well. L_5 is a $T_{\mathbf{P}}$-E_5-ordering with $E_4(x, y) = \exp(-3|x - y|)$ and a $T_{\mathbf{L}}$-E_5-ordering as well. Thus we can define the following fuzzy relations from the fuzzy orderings L_2 (from Example 9), L_3, L_4, and L_5:

$$\mathbf{L}_a = \mathrm{Lex}_{T_{\mathbf{M}},T_{\mathbf{L}}}(L_2, L_2)$$
$$\mathbf{L}_b = \mathrm{Lex}_{T_{\mathbf{L}},T_{\mathbf{L}}}(L_3, L_2)$$
$$\mathbf{L}_c = \mathrm{Lex}_{T_{\mathbf{P}},T_{\mathbf{L}}}(L_4, L_2)$$
$$\mathbf{L}_d = \mathrm{Lex}_{T_{\mathbf{P}},T_{\mathbf{L}}}(L_5, L_5)$$

Theorem 11 then ensures that all these four fuzzy relations are fuzzy orderings with respect to the Łukasiewicz t-norm $T_{\mathbf{L}}$ and $T_{\mathbf{L}}$-equivalences defined as the corresponding Cartesian products. Figure shows cut views of the four lexicographic compositions, where we keep the first argument vector constant (we choose $(x_1, x_2) = (2, 2)$) and plot the value $\mathbf{L}_*((2, 2), (y_1, y_2))$ as a two-dimensional function of y_1 and y_2.

Now the question arises whether the lexicographic composition of two linear fuzzy orderings is again linear. Note that there are several notions of linearity of fuzzy orderings [5]. Let us first consider strong completeness.

Example 13. All fuzzy orderings considered in Examples 9 and 12 were strongly complete. Note, however, that none of the lexicographic compositions defined in Example 12 is strongly complete. To demonstrate that, consider the plots in Figure 3. These two plots show the values

$$\max\big(\mathbf{L}_a((2, 2), (y_1, y_2)), \mathbf{L}_a((y_1, y_2), (2, 2))\big)$$
$$\max\big(\mathbf{L}_d((2, 2), (y_1, y_2)), \mathbf{L}_d((y_1, y_2), (2, 2))\big)$$

as two-dimensional functions of y_1 and y_2. If $\mathbf{L}_a$ and $\mathbf{L}_d$ were strongly complete, these two functions would have to be the constant 1, which is obviously not the case. The same is true for the two other lexicographic compositions $\mathbf{L}_b$ and $\mathbf{L}_c$.

After this negative answer, let us relax the question a bit and attempt the question whether the lexicographic composition of two strongly complete fuzzy orderings is T-linear.

Proposition 14. *Let us make the same assumptions as for Theorem 11. If L_1 and L_2 are strongly complete fuzzy orderings and the residual negation N_T is involutive (i.e. $N_T(N_T(x)) = x$ holds for all $x \in [0, 1]$), then the fuzzy ordering*

$$\mathrm{Lex}_{\tilde{T},T}(L_1, L_2)$$

is T-linear.

Note that Proposition 14 also proves that all the four lexicographic compositions defined in Example 12 are $T_{\mathbf{L}}$-linear.

The proof of Proposition 14 does not work if we do not assume that N_T is an involution. The question arises, of course, whether this condition is not only sufficient, but also necessary. The answer is that this is the case, as the following example demonstrates.

Example 15. Consider a left-continuous t-norm for which a value $z \in]0, 1[$ exists such that $N_T(N_T(z)) \neq z$. Since $N_T(N_T(z)) \geq z$ always holds, we can infer that, in this case,

[2] with $T_{\mathbf{P}}$ denoting the product t-norm

$\mathbf{L}_a$

$\mathbf{L}_b$

$\mathbf{L}_c$

$\mathbf{L}_d$

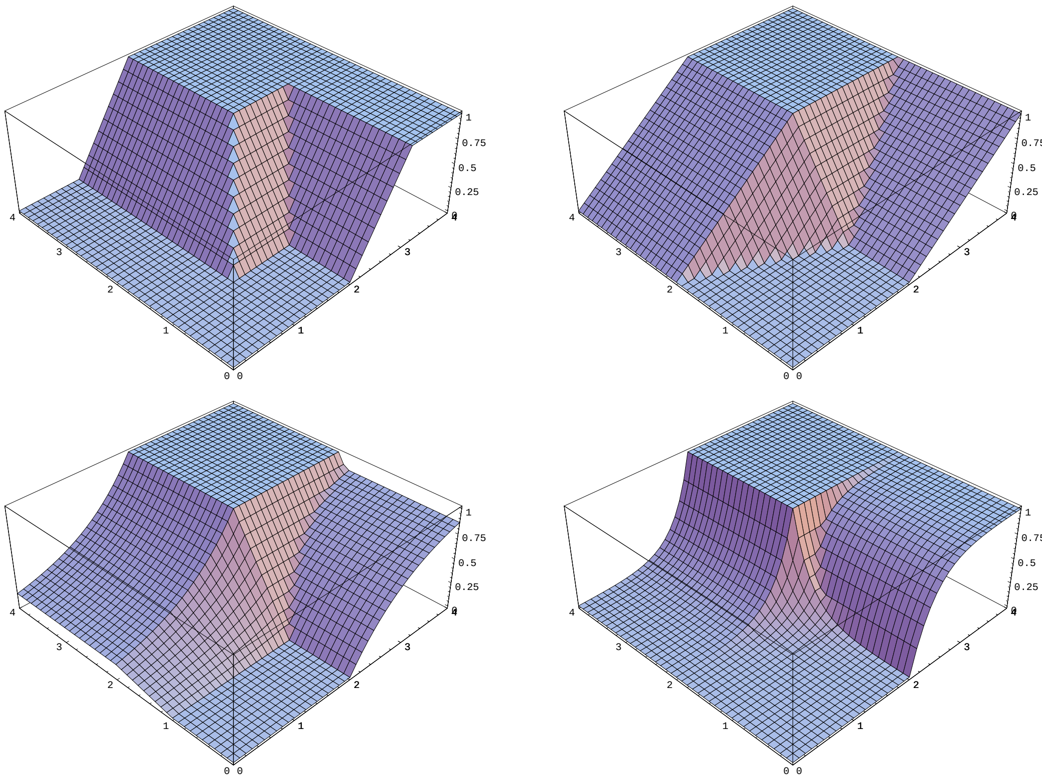

Figure 2: Cut views of the four lexicographic compositions from Example 12

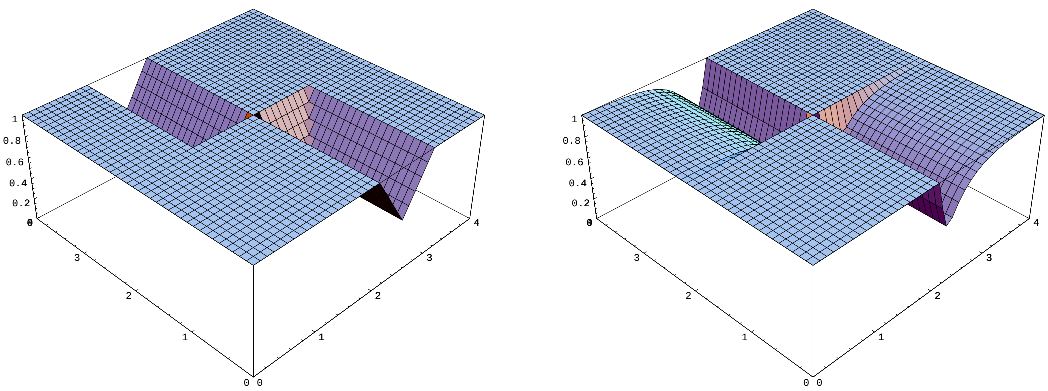

Figure 3: Plots of the functions $\max\big(\mathbf{L}_a((2,2),(y_1,y_2)),\mathbf{L}_a((y_1,y_2),(2,2))\big)$ (left) and $\max\big(\mathbf{L}_d((2,2),(y_1,y_2)),\mathbf{L}_d((y_1,y_2),(2,2))\big)$ (right)

$N_T(N_T(z)) > z$ must hold. Now let us consider two simple strongly complete fuzzy orderings on the sets $X_1 = \{a, b\}$ and $X_2 = \{c, d\}$, respectively:

L_1	a	b
a	1	1
b	z	1

L_2	c	d
c	1	1
d	0	1

Then we can infer the following for any choice of $\tilde{T}$:

$$\begin{aligned}\mathrm{Lex}_{\tilde{T},T}(L_1, L_2)((a, d), (b, c)) &= N_T(L_1(b, a)) \\ &= N_T(z) \\ \mathrm{Lex}_{\tilde{T},T}(L_1, L_2)((b, c), (a, d)) &= L_1(b, a) = z\end{aligned}$$

Hence we obtain that

$$\begin{aligned}N_T\big(\mathrm{Lex}_{\tilde{T},T}&(L_1, L_2)((a, d), (b, c))\big) \\ &= N_T(N_T(z)) > z \\ &= \mathrm{Lex}_{\tilde{T},T}(L_1, L_2)((b, c), (a, d)),\end{aligned}$$

which shows that $\mathrm{Lex}_{\tilde{T},T}(L_1, L_2)$ is not T-linear.

Note that the condition of involutiveness in particular excludes all t-norms without zero divisors. Therefore, lexicographic compositions of non-trivial (i.e. non-crisp) fuzzy orderings with respect to the popular minimum and product t-norms are problematic, if not meaningless. The reason for this is simple. As shown in [4], the only strict fuzzy ordering included in a fuzzy ordering that is strictly greater than zero (e.g. like L_4 and L_5 from Example 12) is the trivial zero relation. When it comes to lexicographic composition, the strict fuzzy ordering induced by the first component relation plays a crucial role. If it vanishes, no meaningful lexicographic composition that preserves linearity properties can be expected. As an example, see Figure 4. It shows a cut view of the fuzzy ordering $\mathrm{Lex}_{T_\mathbf{P},T_\mathbf{P}}(L_5, L_2)$. It is easy to see that $\mathrm{Lex}_{T_\mathbf{P},T_\mathbf{P}}(L_5, L_2)$ is nothing else but the Cartesian product of L_5 and L_2, which is of course not $T_\mathbf{P}$-linear.

The final and most important question is whether the lexicographic composition of two T-linear fuzzy orderings is again T-linear. Strong completeness always implies

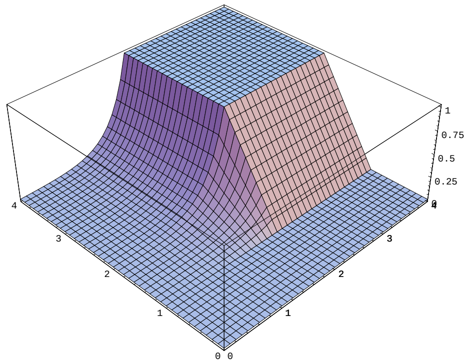

Figure 4: A cut view of $\mathrm{Lex}_{T_\mathbf{P},T_\mathbf{P}}(L_5, L_2)$

T-linearity [5], thus, strongly complete fuzzy orderings are a sub-class of T-linear fuzzy orderings (no matter which T we choose). If the involutiveness of N_T is a necessary condition for meaningful results in Proposition 14, there is no point in considering a t-norm that does not induce an involutive negation any further.

Theorem 16. *Let us again make the same assumptions as for Theorem 11. If L_1 and L_2 are T-linear fuzzy orderings and the residual negation N_T is involutive, then the fuzzy ordering*

$$\begin{aligned}\mathrm{Lex}_{T_\mathbf{M},T}(L_1, L_2)((x_1, x_2), (y_1, y_2)) = \\ \max(\min(L_1(x_1, y_1), L_2(x_2, y_2)), \\ \min(L_1(x_1, y_1), N_T(L_1(y_1, x_1))))\end{aligned}$$

is T-linear.

Obviously, Theorem 16 does not allow any choice of the aggregating t-norm $\tilde{T}$ as in the original construction in Theorem 11, but enforces the choice of the minimum t-norm (i.e. $\tilde{T} = T_\mathbf{M}$). This is not an arbitrary restriction, but a necessary condition, as the following example demonstrates.

Example 17. Consider an arbitrary left-continuous t-norm T that induces a strong negation N_T and assume that $\tilde{T} < T_\mathbf{M}$. Then there exists a $y \in]0, 1[$ such that $\tilde{T}(y, y) < y$. Now let us consider the following two fuzzy relations:

L_1	a	b
a	1	y
b	1	1

L_2	c	d
c	1	y
d	$N_T(y)$	1

It is easy to see that L_1 and L_2 are T-linear fuzzy orderings with respect to T and some T-equivalences (the exact definition of them is not important at this point). Now we can compute:

$$\begin{aligned}
&\mathrm{Lex}_{\tilde{T},T}(L_1, L_2)((a,c),(b,d)) \\
&\qquad = \max(\tilde{T}(y,y), \min(y, N_T(1)) = \tilde{T}(y,y) \\
&\mathrm{Lex}_{\tilde{T},T}(L_1, L_2)((b,d),(a,c)) \\
&\qquad = \max(\tilde{T}(1, N_T(y)), \min(1, N_T(y)) \\
&\qquad = N_T(y)
\end{aligned}$$

If $\mathrm{Lex}_{\tilde{T},T}(L_1, L_2)$ was linear, the following inequality would be fulfilled:

$$\begin{aligned}
&N_T\big(\mathrm{Lex}_{\tilde{T},T}(L_1, L_2)((b,d),(a,c))\big) \\
&\qquad \leq \mathrm{Lex}_{\tilde{T},T}(L_1, L_2)((a,c),(b,d))
\end{aligned}$$

However, we obtain:

$$\begin{aligned}
&N_T\big(\mathrm{Lex}_{\tilde{T},T}(L_1, L_2)((b,d),(a,c))\big) \\
&\qquad = N_T(N_T(y)) = y > \tilde{T}(y,y) \\
&\qquad = \mathrm{Lex}_{\tilde{T},T}(L_1, L_2)((a,c),(b,d))
\end{aligned}$$

Therefore, $\mathrm{Lex}_{\tilde{T},T}(L_1, L_2)$ can never be T-linear if $\tilde{T} < T_{\mathbf{M}}$. This example, therefore, justifies the assumptions of Theorem 16.

5 Conclusion

In this paper, we have introduced an approach to lexicographic composition of similarity-based fuzzy orderings. This construction, in principle, works for all choices of t-norms. However, *the* essential property of lexicographic compositions—that the lexicographic composition of linear orderings is again a linear ordering on the product domain—is only maintained if the underlying t-norm T induces an involutive negation (in particular, including nilpotent t-norms and the nilpotent minimum). This once more confirms the viewpoint that such t-norms are most adequate choices in fuzzy relations theory, fuzzy preference modeling and related fields [4, 5, 8, 10, 21].

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