A Generalized Approach to Fuzzy Orderings and its Applicability in Fuzzy Control

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1 Introduction

Orderings play a fundamental role in the design of fuzzy systems—there might be only a small minority of fuzzy systems in which expressions, such as '*small*', '*medium*', '*large*', etc., do not occur. In general, it is a common way in the design of fuzzy systems to divide the domains of the system variables into a certain number of fuzzy sets with respect to the ordering of the domains.

Beyond this low-level use of orderings, relatively few attention has been paid to the integration of orderings in fuzzy control or related fields at a higher level, although there are some interesting points where the use of orderings could be promising:

- Linguistic approximation
- Definition of ordering-related hedges, such as '*at least*', '*at most*', '*between*', for reducing the size of rulebases
- Rule interpolation

Since we are dealing with vague environments, one would naturally expect fuzzy orderings to be appropriate tools for bridging the gap between orderings and fuzzy sets. First of all, let us recall the well-known fuzzifications of the classical ordering axioms [3, 4, 12, 14].

1.1 Definition Let T be a t-norm and let S be a t-conorm. A binary fuzzy relation R on a crisp set X is called

1. *reflexive* if and only if, for all $x \in X$,
$$R(x, x) = 1,$$
2. T-*antisymmetric* if and only if, for all $x, y \in X$, $x \neq y$ implies
$$T(R(x, y), R(y, x)) = 0,$$
3. T-*transitive* if and only if, for all $x, y, z \in X$,
$$T(R(x, y), R(y, z)) \leq R(x, z)$$
4. S-*complete* if and only if, for all $x, y \in X$, $x \neq y$ implies
$$S(R(x, y), R(y, x)) = 1,$$

5. *strongly S-complete* if and only if, for all $x, y \in X$,

$$S(R(x,y), R(y,x)) = 1.$$

6. *linear* (Zadeh [14]) if and only if, for all $x, y \in X$, either $R(x,y) > 0$ or $R(y,x) > 0$ holds.

It seems to be near at hand to define fuzzy orderings in analogy to the crisp case (i.e., Zadeh's definition [14] but for arbitrary t-norms instead the minimum t-norm).

1.2 Definition A reflexive, T-antisymmetric, and T-transitive binary fuzzy relation is called fuzzy (partial) ordering.

Beside that, there are considerably many other definitions [4, 9, 12], most of them omitting some of the classical axioms. Let us examine in more detail why this could be desirable.

Consider, for example, the problem how to define a linear/complete fuzzy ordering of the real numbers which should be a fuzzification of the linear ordering. A natural requirement on such an ordering would be the following monotonicity:

$$\forall x, y, z \in \mathbb{R}: \quad y \leq z \Longrightarrow R(x,y) \leq R(x,z) \tag{1}$$

This is, more or less, a formulation of compatibility between R and $\leq$, stating that, for a fixed x, the degree of being greater or equal than x is an increasing function. Let us see what happens if we assume such an R to fulfill the axioms of Definition 1.2. First of all, for any arbitrary but fixed x, reflexivity implies that

$$R(x,x) = 1.$$

Then, taking (1) into account, we obtain that

$$R(x,y) = 1 \qquad \forall y \geq x.$$

Finally, antisymmetry implies that

$$R(x,y) = 0 \qquad \forall y \leq x,$$

and we have shown that only crisp orderings can fulfill both the axioms of Definition 1.2 and (1). Note that linearity/completeness has not even been taken into acount. Even if reflexivity is omitted, which is not so unusual [4, 12], linearity/completeness and antisymmetry together imply that R must be discontinuous on the main diagonal.

The author is deeply convinced that simply omitting axioms does not solve the problem sufficiently, since the axioms of classical orderings proved to be good for centuries; every single one has its own justification—omitting just opens the field for arbitrariness.

The above definition of T-antisymmetry is equivalent to

$$T(R(x,y), R(y,x)) \leq E_{=}(x,y), \tag{2}$$

where $E_{=}$ is the crisp equality

$$E_{=}(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

One immediately sees that this is indeed a sound fuzzification of the classical axiom of antisymmetry

$$(x \leq y \land y \leq x) \Longrightarrow x = y,$$

where $\leq$ is replaced by a fuzzy ordering. The crisp equality on the right hand side, however, remains unfuzzified. In fact, this seems to contradict to the nature of vague environments. Thinking of fuzzy orderings as models for expressions like "*approximately smaller/greater or equal*", one immediately observes that requiring crisp equality seems to be unnaturally strong.

The key idea for overcoming all the above difficulties is to replace the crisp equality in (2) by a fuzzy concept of equality. A rather common tool for that are *fuzzy equivalence relations* (other terms are similarity relation [14], indistinguishability relation [13], and fuzzy equality [8, 10]; we will use the term fuzzy equivalence relation since it reveals in the best way that the definition is, more or less, a fuzzification of the axioms of classical equivalence relations).

1.3 Definition Let T be a t-norm. Then a mapping $E : X^2 \to [0,1]$, where X is an arbitrary crisp set, is called *fuzzy equivalence relation* with respect to T, if and only if it satifies the following three properties:

$$\begin{array}{lll} \forall x \in X: & E(x,x) = 1 & \text{(Reflexivity)} \\ \forall x,y \in X: & E(x,y) = E(y,x) & \text{(Symmetry)} \\ \forall x,y,z \in X: & T(E(x,y), E(y,z)) \leq E(x,z) & (T\text{-Transitivity}) \end{array}$$

A fuzzy equivalence relation is called separated if and only if the stricter reflexivity holds:

$$\forall x,y \in X: \quad E(x,y) = 1 \Longleftrightarrow x = y$$

1.4 Definition Let T be a t-norm and let E be a fuzzy equivalence relation on a set X with respect to T. Then a mapping $L : X^2 \to [0,1]$ is called a *fuzzy ordering* with respect to T and E, if and only if it satifies the following three properties:

$$\begin{array}{lll} \forall x, y \in X: & L(x,y) \geq E(x,y) & (E\text{-Reflexivity}) \\ \forall x, y \in X: & T(L(x,y), L(y,x) \leq E(x,y) & (T, E\text{-Antisymmetry}) \\ \forall x, y, z \in X: & T(L(x,y), L(y,z)) \leq L(x,z) & (T\text{-Transitivity}) \end{array}$$

L is called a *linear fuzzy ordering* if for every pair (x,y) either $L(x,y) = 1$ or $L(y,x) = 1$ holds.

Obviously, not only the definition of antisymmetry, also reflexivity and linearity have been modified. The reasons are the following:

1. Since, in the crisp and in the fuzzy case (cf. Definition 1.1), the meaning of reflexivity is that equality implies being in relation, there is no reason to violate this relationship here. The only exception is that the crisp equality has to be replaced by the fuzzy equivalence as in the modified definition of antisymmetry.

2. If one thinks of fuzzy orderings as fuzzifications of crisp orderings, it seems to be justified to require a degree of 1 where it is known with certainty that the two values are in relation already in the crisp sense.

1.5 Lemma *Some basic properties:*

1. *Every fuzzy equivalence relation E with respect to a t-norm T is a fuzzy ordering with respect to T and itself.*

2. *Every crisp ordering is a fuzzy ordering with respect to any t-norm and the crisp equality.*

3. *If L is a fuzzy ordering, then the inverse $G(x,y) = L(y,x)$ is also a fuzzy ordering with respect to the same t-norm and fuzzy equivalence relation.*

4. *Every fuzzy ordering is a fuzzy ordering with respect to any weaker t-norm and the same fuzzy equivalence relation.*

5. *Every fuzzy ordering in the sense of Definition 1.2 is a fuzzy ordering in the new sense with respect to T and the crisp equality.*

The last point illustrates in which way the new approach generalizes the previous one of Definition 1.2.

2 Constructions and Representations

It is easy to prove that, for every crisp relation $\trianglelefteq$, which is reflexive and transitive (often called preordering), the symmetric kernel

$$x \bigtriangleup y \Longleftrightarrow (x \trianglelefteq y \wedge x \trianglerighteq y)$$

is an equivalence relation. The fuzzy analogon has been proven by Valverde [13]. Moreover, there is a strong connection to fuzzy orderings.

<u>2.1 Theorem</u> *Let T be an arbitrary t-norm and let L be a reflexive and T-transitive fuzzy relation (often called a fuzzy preorder). Then L is a fuzzy ordering with respect to T and the fuzzy equivalence relation*

$$E(x,y) = T(L(x,y), L(y,x)).$$

From this point of view, the new approach seems to be the hidden removal of the antisymmetry axiom. One should, however, not neglect that there is auxiliary information—the fuzzy equivalence relation—which allows to control the behavior and influence of antisymmetry. The problem is rather the generality of fuzzy equivalence relations than the generality of fuzzy orderings.

The next theorem shows how to define a fuzzy ordering on a product space provided that there are fuzzy orderings of each component.

<u>2.2 Theorem</u> *Let $X_1, \dots, X_n$ be crisp sets and let T be an arbitrary t-norm. If $(L_1, \dots, L_n)$ and $(E_1, \dots, E_n)$ are families of fuzzy relations such that, for all $i \in \{1, \dots, n\}$,*

1. *L_i and E_i are binary fuzzy relations defined on X_i,*
2. *E_i is a fuzzy equivalence relation on X_i with respect to T,*
3. *L_i is a fuzzy ordering on X_i with respect to E_i and T,*

then the fuzzy relation

$$\begin{array}{rccl} \tilde{L}: & (X_1 \times \cdots \times X_n)^2 & \longrightarrow & [0,1] \\ & ((x_1, \dots, x_n), (y_1, \dots, y_n)) & \longmapsto & \mathop{T}\limits_{i=1}^{n} L_i(x_i, y_i) \end{array}$$

is a fuzzy ordering with respect to T and the fuzzy equivalence relation

$$\tilde{E}((x_1, \dots, x_n), (y_1, \dots, y_n)) = \mathop{T}\limits_{i=1}^{n} E_i(x_i, y_i).$$

As already promised, the new model is able to fuzzify crisp linear orderings in a way, such that condition (1) is fulfilled. After one prerequisite, we can show how.

2.3 Definition Let $\lesssim$ be a crisp ordering on X and let E be a fuzzy equivalence relation on X. E is called *compatible with* $\lesssim$, if and only if the following implication holds for all $x, y, z \in X$:

$$x \lesssim y \lesssim z \Longrightarrow (E(x,z) \leq E(y,z) \wedge E(x,z) \leq E(x,y)). \tag{3}$$

Property (3) can be interpreted as follows: For a fixed $x \in X$, the mapping $f(y) := E(x,y)$ is increasing for $y \lesssim x$ and decreasing for $y \gtrsim x$.

2.4 Theorem *Let L be a binary fuzzy relation on X and let E be a fuzzy equivalence relation E with respect to an arbitrary t-norm T. Then the following two statements are equivalent:*

1. *L is a linear fuzzy ordering on X with respect to T and E.*
2. *There exists a linear ordering $\lesssim$ the relation E is compatible with such that L can be represented as follows:*

$$L(x,y) = \begin{cases} 1 & \text{if } x \lesssim y \\ E(x,y) & \text{otherwise} \end{cases} \tag{4}$$

Theorem 2.4 states that linear fuzzy orderings are uniquely characterized as unions of crisp linear orderings and fuzzy equivalence relations. In this sense, linear fuzzy orderings are straightforward fuzzifications of crisp linear orderings, where the fuzzy component can be attributed to a fuzzy equivalence relation.

3 Hulls and Hedges

3.1 Definition Let R be an arbitrary fuzzy relation on a domain X and let A be a fuzzy subset of X. Then the hull of A with respect to R and a t-norm T is defined as

$$H_R(A)(x) = \sup\{T(A(y), R(y,x) \mid y \in X\}.$$

If R is a fuzzy equivalence relation on X with respect to T, the hull is often called *extensional hull*, which we will denote with the symbol $\mathrm{EXT}(A)$. If R is a fuzzy ordering, the symbol $ATL(A)$ (*'at least A'*) will be used for the hull. Moreover, for the hull with respect to the inverse fuzzy ordering $G(x,y) = L(y,x)$, the symbol $ATM(A)$ (*'at most A'*) will be used.

3.2 Definition For a fuzzy subset A of a domain X, which is equipped with an ordering $\lesssim$, we define the following operators (left-to-right and right-to-left continuations, convex hull):

$$\begin{aligned}
\mathrm{LTR}(A)(x) &= \sup\{\mu_A(y) | y \lesssim x\} \\
\mathrm{RTL}(A)(x) &= \sup\{\mu_A(y) | x \lesssim y\} \\
\mathrm{CVX}(A)(x) &= \min(\mathrm{LTR}(A)(x), \mathrm{RTL}(A)(x))
\end{aligned}$$

It is not difficult to prove that LTR and RTL are hull operators with respect to the following crisp orderings:

$$\mathrm{LTR}(A) = H_{R_1}(A) \quad \text{where} \quad R_1(x,y) = \begin{cases} 1 & \text{if } x \lesssim y \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{RTL}(A) = H_{R_2}(A) \quad \text{where} \quad R_2(x,y) = \begin{cases} 1 & \text{if } x \gtrsim y \\ 0 & \text{otherwise} \end{cases}$$

3.3 Theorem *Provided that L is a linear fuzzy ordering of a domain X with respect to a fuzzy equivalence relation E and a t-norm T, for every fuzzy subset A in X, the equalities*

$$\begin{aligned}
\mathrm{ATL}(A) &= \mathrm{LTR}(\mathrm{EXT}(A)) &= \mathrm{EXT}(\mathrm{LTR}(A)) \\
\mathrm{ATM}(A) &= \mathrm{RTL}(\mathrm{EXT}(A)) &= \mathrm{EXT}(\mathrm{RTL}(A)) \\
\mathrm{ECX}(A) &= \mathrm{CVX}(\mathrm{EXT}(A)) &= \mathrm{EXT}(\mathrm{CVX}(A))
\end{aligned}$$

hold, where $\mathrm{ECX}(A)$ *(extensional convex hull) is defined in the following way:*

$$\mathrm{ECX}(A)(x) = \min(\mathrm{ATL}(A)(x), \mathrm{ATM}(A)(x))$$

We have seen that the ordering-related hedges "*at least*" and "*at most*" can be defined as hulls with respect to a fuzzy ordering and its inverse, respectively. Moreover, these definitions can be used to define other hedges. Some examples, where $\complement$ is the complement with respect to some generalized negation n and S is a t-conorm such that (T, S, n) is a De-Morgan triple [4]:

1. Strictly greater than A:

$$\mathrm{SGT}(A) = \mathrm{ATL}(A) \cap_T \complement(\mathrm{ATM}(A))$$

2. Strictly less than A:

$$\mathrm{SLS}(A) = \mathrm{ATM}(A) \cap_T \complement(\mathrm{ATL}(A))$$

3. Within A:

$$\mathrm{WIT}(A) = \mathrm{ECX}(A) \cap_T \complement(\mathrm{EXT}(A))$$

4. Convex closure of A and B:

$$\mathrm{CCL}(A,B) = (\mathrm{ATL}(A) \cap_T \mathrm{ATM}(B)) \cup_S (\mathrm{ATM}(A) \cap_T \mathrm{ATL}(B))$$

5. Between A and B:

$$\mathrm{BTW}(A,B) = (\mathrm{SGT}(A) \cap_T \mathrm{SLS}(B)) \cup_S (\mathrm{SLS}(A) \cap_T \mathrm{SGT}(B))$$

4 Ordering Fuzzy Sets

One of the most important offsprings of the new theory is a general way how to define (pre)orderings of fuzzy sets of an arbitrary linearly ordered set with the possibility to incorporate indistinguishability. Orderings of fuzzy sets are especially important for linguistic approximation and rule interpolation.

4.1 Theorem *If L is a linear fuzzy ordering on a domain X, then the following binary relation, which is defined on $\mathcal{F}(X)$, the set of fuzzy subsets of X,*

$$A \lesssim_L B \iff \mathrm{ATL}(A) \supset \mathrm{ATL}(B) \wedge \mathrm{ATM}(A) \subset \mathrm{ATM}(B) \qquad (5)$$

is reflexive and transitive, where the symmetric kernel of $\lesssim_L$ is uniquely characterized as

$$A \sim B \iff \mathrm{ECX}(A) = \mathrm{ECX}(B). \qquad (6)$$

4.2 Remark According to the considerations at the beginning of Section 2, Equation (6) states to which degree the relation $\lesssim$ is antisymmetric. In particular, we obtain that $\lesssim$ is indeed an ordering on the subclass of extensional convex fuzzy subsets.

If no indistinguishability is taken into account, i.e., if L is a crisp linear ordering, (5) is equivalent to

$$A \lesssim B \iff \mathrm{LTR}(A) \supset \mathrm{LTR}(B) \wedge \mathrm{RTL}(A) \subset \mathrm{RTL}(B).$$

Under some weak continuity conditions, this is equivalent to an ordering of fuzzy sets proposed in [11]. If we additionally restrict to fuzzy numbers with one-elementary kernels, both concepts are equivalent to the ordering defined in [15]. In this sense, the above definition is a generalization of some already existing approaches to orderings of fuzzy sets.

The above preordering of fuzzy sets allows us to compare two arbitrary fuzzy sets. Unlike other approaches, where the restriction to a special class of fuzzy sets is made at the beginning [1, 2, 15], this approach can be applied to any different kind of fuzzy subsets of a linearly ordered domain. In particular, no other assumptions about the structure of the space X (e.g., completeness, restriction to real numbers or intervals, etc.) are made. The only restriction is that this "ordering" cannot distinguish between fuzzy sets with equal extensional convex hulls.

For many problems, it can be sufficient to treat fuzzy sets with the same extensional convex hull as equivalent. If, for what reasons ever, one is interested in a fully antisymmetric ordering in the crisp sense, it is sufficient to find orderings of all equivalence classes. Then, by applying lexicographic composition, an ordering of fuzzy sets is obtained, where the coarse comparison is carried out by the above preordering.

In the next step, we can define a fuzzy ordering of fuzzy sets as a fuzzification of the crisp ordering defined by (5). Let us consider the following equivalent formulation:

$$\forall x \in X: \quad \mu_{\mathrm{ATL}(A)}(x) \geq \mu_{\mathrm{ATL}(B)}(x) \wedge \mu_{\mathrm{ATM}(A)}(x) \leq \mu_{\mathrm{ATM}(B)}(x).$$

In order to fuzzify this formula, we have to choose fuzzifications for the quantifier, the conjunction, and the inclusions. For the universal quantifier, the infimum is the natural candidate [7], as well as the underlying t-norm T is an obvious candidate for the conjunction. Of course, inclusions are closely related to implications. So far, there are different concepts of fuzzy implications. Among all these, only so-called residual implications are suitable for the problem of fuzzifying inclusions. We will see later why.

4.3 Definition Let T be an arbitrary t-norm. A binary operation on $[0,1]$ is called *residuum* (R-implication, Φ-operator) of T if and only if for all $x, y, z \in [0,1]$ the equivalence

$$T(x,y) \leq z \Longleftrightarrow x \leq R(y,z)$$

holds.

Before turning to fuzzy orderings of fuzzy sets again, we need a few elementary properties of residual impliations [5, 6, 7].

4.4 Theorem *For any continuous t-norm T, there exists a unique residuum $\vec{T}$ which is given as*

$$\vec{T}(x,y) = \sup\{z \in [0,1] \mid T(x,z) \leq y\}.$$

4.5 Theorem *For any continuous t-norm T, the following two properties hold:*

1. $\forall x, y \in [0,1]: \quad x \leq y \Leftrightarrow \vec{T}(x,y) = 1$
2. $\forall x, y, z \in [0,1]: \quad T(\vec{T}(x,y), \vec{T}(y,z)) \leq \vec{T}(y,z)$

4.6 Remark

1. The inequality 2. in Theorem 4.5 is known as the transitivity of the residual implication. This property, which, among different types of fuzzy implications, can only be guaranteed for residua, ensures that inclusions are always T-transitive.

2. Directly following from Theorems 2.1 and 4.5, for any continuous t-norm T, the residuum $\vec{T}$ is a fuzzy ordering of $[0,1]$ with respect to T and its so-called biimplication

$$\overset{\leftrightarrow}{T}(x,y) = T(\vec{T}(x,y), \vec{T}(y,x)).$$

Now we can define a fuzzy relation as a straightforward fuzzification of $\lesssim_L$. However, we will have to prove the compatibility with the crisp case:

$$\begin{array}{rcl} \tilde{L}: \ \mathcal{F}(X) \times \mathcal{F}(X) & \to & [0,1] \\ (A,B) & \mapsto & T\big(\inf\limits_{x\in X} \vec{T}(\mu_{\mathrm{ATL}(B)}(x), \mu_{\mathrm{ATL}(A)}(x)), \\ & & \inf\limits_{x\in X} \vec{T}(\mu_{\mathrm{ATM}(A)}(x), \mu_{\mathrm{ATM}(B)}(x))\big) \end{array} \tag{7}$$

First of all, the following lemma shows under which conditions this relation conincides with the crisp case.

4.7 Lemma *For all fuzzy subsets A, B of X, the following holds:*

$$\tilde{L}(A,B) = 1 \quad \Longleftrightarrow \quad A \lesssim_L B$$

Moreover, as promised, $\tilde{L}$ is even a fuzzy ordering.

4.8 Theorem *The binary fuzzy relation $\tilde{L}$ as defined in (7) is a fuzzy ordering with respect to T and the fuzzy equivalence relation*

$$\begin{aligned} E_{\tilde{L}}(A,B) = T\big(&\inf_{x\in X} \vec{T}(\mu_{\mathrm{ATL}(B)}(x), \mu_{\mathrm{ATL}(A)}(x)), \\ &\inf_{x\in X} \vec{T}(\mu_{\mathrm{ATM}(A)}(x), \mu_{\mathrm{ATM}(B)}(x)), \\ &\inf_{x\in X} \vec{T}(\mu_{\mathrm{ATL}(A)}(x), \mu_{\mathrm{ATL}(B)}(x)), \\ &\inf_{x\in X} \vec{T}(\mu_{\mathrm{ATM}(B)}(x), \mu_{\mathrm{ATM}(A)}(x))\big). \end{aligned} \tag{8}$$

Again according to the discussions in connection with Theorem 2.1, we have to check in which way (8) can be considered as a reasonable measure of similarity.

<u>4.9 Theorem</u> *With the above settings, the following holds for all fuzzy subsets $A, B \in \mathcal{F}(X)$:*

$$E_{\tilde{L}}(A, B) \leq \inf_{x \in X} \overset{\leftrightarrow}{T}\big(\mu_{\mathrm{ECX}(A)}(x), \mu_{\mathrm{ECX}(B)}(x)\big)$$

Moreover, $E_{\tilde{L}}$ is separated on the subclass of extensional convex fuzzy sets:

$$E_{\tilde{L}}(A, B) = 1 \iff \mathrm{ECX}(A) = \mathrm{ECX}(B)$$

Finally, taking Lemma 4.7 and Theorem 4.9 into account, we see that $\tilde{L}$ indeed provides a proper fuzzification of the ordering $\lesssim_L$.

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