A New Approach to Fuzzy Orderings

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Fuzzy orderings can be useful in fuzzy control for two purposes:

1. Definition of ordering-related hedges, such as 'at least' or 'approximately between'. Such constructs can help to keep rulebases compact and surveyable.
2. Examination of semantics of fuzzy sets/partitions (e.g., linguistic approximation).

Unfortunately, due to very strict antisymmetry criteria, previous approaches [4, 2] turned out to be useless for fuzzy control purposes.

The following definition, which is based on the concept of fuzzy equalities [1, 3], overcomes these disadvantages by relaxing the antisymmetry axiom:

Definition: Let T be a t-norm and let E be a fuzzy equality with respect to T. Then a mapping $L : X^2 \to [0,1]$ is called a *fuzzy ordering* with respect to T and E, if and only if it satifies the following three axioms:

$$\begin{array}{lll} \forall x,y \in X: & L(x,y) \geq E(x,y) & \text{(Reflexivity)} \\ \forall x,y \in X: & T(L(x,y), L(y,x) \leq E(x,y) & \text{(Antisymmetry)} \\ \forall x,y,z \in X: & T(L(x,y), L(y,z)) \leq L(x,z) & \text{(Transitivity)} \end{array}$$

L is called a *linear fuzzy ordering* if for every pair (x,y) either $L(x,y)=1$ or $L(y,x)=1$ holds.

Besides a lot of basic properties of crisp orderings which still hold in the fuzzy case, it is worth to mention that linear fuzzy orderings can always be represented as compositions of crisp linear orderings and fuzzy equalities.

As already mentioned, fuzzy orderings can be used to define ordering-related hedges.

Definition: Let L be a fuzzy ordering as defined above and let A be a fuzzy subset of X. Then the fuzzy set $\mathrm{ATL}(A)$ ('at least A') can be defined as follows:

$$\mu_{\mathrm{ATL}(A)}(x) = \sup\{T(\mu_A(y), L(y,x)) \mid y \in X\}$$

Theorem: *If L is a linear fuzzy ordering, for each fuzzy set A, the equation*

$$\mathrm{ATL}(A) = \mathrm{LTR}(\mathrm{EXT}(A)) = \mathrm{EXT}(\mathrm{LTR}(A))$$

holds, where $\mathrm{EXT}(A)$ is the extensional hull of A and $\mathrm{LTR}(A)$ is the so-called left-to-right continuation of A:

$$\mu_{\mathrm{LTR}(A)}(x) = \sup\{\mu_A(y) \mid y \leq x\}$$

The hedge ATL can be used to define a lot of other ordering-related hedges which can help to reduce size and complexity of rulebases. Moreover, such hedges can be used to define orderings of fuzzy sets, which can be helpful for the investigation of the semantics of fuzzy sets/partitions.

References

[1] U. Höhle. Fuzzy equalities and indistinguishability. In *Proc. EUFIT'93*, volume 1, pages 358–363, 1993.

[2] A. Kaufmann. *Introduction to the Theory of Fuzzy Subsets*, volume I. Academic Press, New York, 1975.

[3] R. Kruse, J. Gebhardt, and F. Klawonn. *Foundations of Fuzzy Systems*. John Wiley & Sons, New York, 1994.

[4] L. A. Zadeh. Similarity relations and fuzzy orderings. *Inform. Sci.*, 3:177–200, 1971.