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Begutachter: Univ.-Prof. Dr. Erich Peter Klement
Univ.-Prof. Dr. Siegfried Gottwald

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Dedicated to the memory of
Gerald Scheinmayr

Abstract

Many fuzzy systems in real-world applications make implicit use of two fundamental concepts—similarity and ordering. For both of them, formulations in the framework of fuzzy relations have been proposed already in the early days of fuzzy set theory. While similarity relations have turned out to be very useful tools for the interpretation of fuzzy partitions and fuzzy controllers, the utilization of fuzzy orderings in more applied areas is still lying far behind. The main objective of this thesis is to find reasons for this missing link and, consequently, to present and investigate an alternative approach to fuzzy orderings which overcomes the problems in terms of applicability.

First of all, the need for expressing orderings in vague environments is motivated with several examples. After providing the necessary preliminaries from the theory of fuzzy sets and relations, we turn to a critical view on the existing approaches to fuzzy orderings. By the help of three case studies, the definition of fuzzy antisymmetry turns out to be the crucial point. Resting upon this discovery, a generalization of fuzzy orderings is presented which also takes the strong relationship between similarity and ordering into account. The key idea is to replace the crisp equality in the definition of reflexivity and antisymmetry by a similarity relation.

The remaining thesis is devoted to three topics. Firstly, constructions, representations, and characterizations of the new class of fuzzy orderings and their dual relations are studied in detail. Secondly, we investigate properties and representations of hulls with respect to fuzzy orderings which can be particularly useful for applications. Finally, a general framework for ordering fuzzy sets is introduced which can be applied to any domain for which a crisp or fuzzy ordering is known.

Zusammenfassung

Viele, wenn nicht alle Fuzzy Systeme in tatsächlichen Implementierungen bedienen sich zweier grundlegender Konzepte — Ähnlichkeit bzw. Ununterscheidbarkeit und Ordnung. Beide waren schon in der Anfangszeit von Fuzzy Logik und Fuzzy Systemen Gegenstand von Untersuchungen. Die Formulierung von Ähnlichkeitsbeziehungen als Fuzzy Relationen hat sich dabei als besonders nützlich für die Interpretation von Fuzzy Partitionen und Fuzzy Reglern herausgestellt. Fuzzy Ordnungen hingegen haben sich bislang noch nicht als für Anwendungen tauglich erweisen können. Das Ziel der vorliegenden Arbeit ist, einen neuen Ansatz zu Fuzzy Ordnungen zu entwickeln, der die Probleme hinsichtlich Anwendbarkeit überwindet.

Zu diesem Zweck wird zunächst die Notwendigkeit von unscharfen Ordnungsbegriffen anhand von Beispielen motiviert. Nach Bereitstellung der notwendigen Grundlagen aus der Theorie der Fuzzy Mengen und Relationen wenden wir uns der kritischen Betrachtung der bestehenden Ansätze zu Fuzzy Ordnungen zu, wobei sich herausstellt, daß hauptsächlich die verallgemeinerte Definition der Antisymmetrie für die Schwierigkeiten verantwortlich ist. Fußend auf dieser Erkenntnis wird ein alternativer Ansatz vorgestellt, der auch der engen Beziehung zwischen Ordnung und Ununterscheidbarkeit Rechnung trägt. Die Schlüsselidee ist dabei, die scharfen Gleichheiten in den verallgemeinerten Definitionen von Reflexivität und Antisymmetrie durch Ähnlichkeitsrelationen zu ersetzen.

Die verbleibende Arbeit ist drei Themen gewidmet. Erstens werden Konstruktionen und Darstellungen der neuen Klasse von Fuzzy Ordnungen und ihrer dualen Relationen untersucht. Im weiteren werden die von Fuzzy Ordnungen induzierten Hüllenoperatoren untersucht, die für Anwendungen besonders nützlich sein könnten. Abschließend erfolgen Untersuchungen über einen allgemeinen Ansatz zu Ordnungen von Fuzzy Mengen, der lediglich von der Vorgabe einer scharfen oder Fuzzy Ordnung ausgeht.

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Contents

Chapter 1

Introduction

The most important novelty of L. A. Zadeh's work was neither that he introduced a kind of many-valued logic nor that he suggested it as a model of uncertainty. Both had been done much earlier by Gödel [22], Łukasiewicz [45], Menger [48], and others. The brilliant idea was to utilize what Zadeh called "fuzzy sets" as mathematical models of linguistic expressions which cannot be represented in the framework of classical binary logic and set theory in a natural way. In the introduction of his epoch-making article on fuzzy sets [72], he writes:

> *"More often than not, the classes of objects encountered in the real physical world do not have precisely defined criteria of membership. [. . .] Yet, the fact remains that such imprecisely defined "classes" play an important role in human thinking, particularly in the domains of pattern recognition, communication of information, and abstraction."*

Of course, this applicability also requires an inference mechanism which is at least an approximate model of the way humans make decisions and conclusions employing imprecisely defined linguistic expressions. After the introduction of such methods [73, 75], it took about twenty years for this new paradigm of "fuzzy systems" to become widely accepted. But more than that, it was then, in fact, a tremendous success which started with well-selling applications in consumer goods implemented by Japanese engineers. The reasons for this late, but, nevertheless, enormous triumph of fuzzy systems include the following points:

1. The main difference between fuzzy systems and other control or decision support systems is that they are parameterized in an interpretable

way—by means of rules consisting of linguistic expressions. Fuzzy systems, therefore, allow rapid prototyping as well as easy maintenance and adaptation.

2. Closely related to the previous point, fuzzy systems provide simple and robust solutions—an advantage which is particularly important for applications in mass products, where production costs are especially important.

3. Fuzzy systems offer completely new opportunities to deal with processes for which only a linguistic description is available and, as a result, to achieve a robust, secure, and reproducible automation of such tasks.

4. Even if conventional strategies can be employed, reformulating a system's actions by means of linguistic rules can lead to a deeper understanding of its behavior.

Undoubtedly, interpretability is the red thread common to all the above four points. On the other hand, not everything, which is a fuzzy system from the formal point of view, is really interpretable by humans. In this manner, the study of interpretability is a fundamental and crucial task in the theory of fuzzy sets and systems. Two concepts play an essential role when it concerns interpretability—similarity and ordering.

It is a common property of human thinking to take gradual similarity into account. In fact, this is exactly the point where binary logic fails to be an appropriate model and, in this sense, the key motivation for establishing fuzziness. Consider, for instance, how to define the set of tall people. Specifying a sharp limit, e.g. 180cm, leads to unnatural preciseness. While a person of 179.9cm would be classified as not tall, somebody of 180.1cm would be classified as tall, although it is not even possible for a human to distinguish between the two without taking a measuring tape. Allowing gradual transitions between the two classes, as fuzzy sets do, solves this problem in a very elegant way. Adopting this point of view, intuition suggests that gradual similarity is, in some sense, an inherent component of fuzziness.

On the other hand, in almost all applications, the domains of the input variables, at least in the case of real intervals, are divided into a certain number of fuzzy subsets by means of the underlying ordering—there might be only a small minority of fuzzy systems in which expressions, such as *'small'*, *'medium'*, or *'large'*, do not occur. So, it is easy to observe that orderings are essential ingredients of fuzzy systems as well. Not very much attention, however, has been paid to the integration of orderings into fuzzy systems on a higher level, although there are considerably many points where orderings, maybe together with similarity, can be helpful:

1. A problem common to most of the methods for refining or designing fuzzy systems from example data is that they can yield results which are fuzzy systems formally, but no longer interpretable. Since similarity and ordering are two key concepts in the design of interpretable fuzzy systems, it would be a promising approach to employ them also for defining criteria of interpretability and, one step further, for constructing parametrizations which only allow interpretable settings at all.

2. As orderings are almost always used for the specification of fuzzy sets, it seems to be natural to apply them also to the inverse procedure called linguistic approximation, where a linguistic expression has to be found which describes a previously unknown fuzzy set. Most approaches to linguistic approximation are based on comparisons with a given library of fuzzy sets. Orderings could provide a way to find an interpretation of the semantics of a fuzzy set even if no sample sets are taken into account—possibly in connection with existing similarity-based methods [29, 35].

3. Most implementations of fuzzy systems use tables for representing their rule bases. It is trivial to see that, for such systems, the number of rules grows exponentially with the number of variables—a fact which can be regarded as a serious limitation in terms of surveyability and interpretability. Among other measures, it could be a promising approach to use ordering-based operators, such as *'at least'*, *'at most'*, or *'between'*, for grouping neighboring rules with the same consequents in order to reduce their overall number.

4. Sometimes, when experts or automatic tuning procedures only provide an incomplete description of a fuzzy rule base, it can still be necessary to obtain a conclusion even if an observation does not match any antecedent in the rule base [38]. Moreover, it is considered as another opportunity for reducing the size of a rule base to store only some representative rules and to interpolate between them [39]. In any case, it is indispensable to have criteria for determining between which rules the interpolation should take place. Beside distance, orderings play a fundamental role in this selection. Furthermore, as an alternative to distance-based methods [39], it is possible to fill the gap between the

antecedents of two rules using a fuzzy concept of *'between'*, which leads us to the ordering-based operators mentioned above.

For both similarity and ordering, formulations in the framework of fuzzy relations have been proposed already in the early days of fuzzy set theory [74]. So-called similarity relations have turned out to be useful tools for investigating the semantics and interpretability of fuzzy sets [29, 35], at least up to the point where the structure of the underlying space, i.e. the ordering, should be taken into account. Fuzzy orderings, however, could not yet prove applicability to any of the above points.

The aim of this thesis is to discuss the counter-intuitive properties of fuzzy orderings and, in the following, to define a generalization which overcomes these problems.

Chapter 2

Preliminaries

This chapter is intended to provide the mathematical apparatus for the following studies of fuzzy orderings and their properties in a comprehensive way. Unless stated otherwise, only original results will be proved (mainly in 2.1.2 and 2.4.2). For the proofs of the remaining reproduced results, we will refer to the literature. In any case, the reader is assumed to be familiar with the basics of equivalence relations, orderings, and lattice theory [6, 41].

2.1 Fuzzy Sets

2.1.1 Basic Notions

2.1. Definition. Let X be an arbitrary non-empty set. A *fuzzy subset* A of X is uniquely represented by its membership function

$$\mu_A : X \longrightarrow [0, 1]$$

where the value $\mu_A(x)$ is interpreted as the degree to which the value x is contained in A. The set of all fuzzy subsets of X is called *fuzzy power set* of X and denoted by the symbol $\mathcal{F}(X)$.

2.2. Definition. Let A, B be two fuzzy subsets of X.

1. A and B are called *equal*, denoted $A = B$, if and only if their membership functions coincide.
2. A is called *subset* of B, short $A \subseteq B$, if and only if, for all $x \in X$,

   $$\mu_A(x) \leq \mu_B(x).$$

   Consequently, B is called *superset* of A.

The unit interval $[0, 1]$ is the domain of truth values we will consider in the following. We restrict ourselves to this domain, although most of the following results hold analogously even if more general domains are used [23, 27, 34]. The reasons are simplicity and the fact that fuzzy systems applications, which we are particularly interested in, almost exclusively use the unit interval.

Every ordinary set M is uniquely determined by its characteristic function

$$\chi_M(x) = \begin{cases} 1 & \text{if } x \in M, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, it can also be regarded as a fuzzy set. As a consequence, the crisp power set $\mathcal{P}(X)$ can be embedded into $\mathcal{F}(X)$. In order to explicitly distinguish between fuzzy and ordinary sets, we will often use the term "crisp" for non-fuzzy.

2.3. Definition. Let A be a fuzzy subset of a domain X.

1. The *height* of A is defined as

$$\text{height}(A) = \sup\{\mu_A(x) \mid x \in X\}.$$

2. The *support* of A is defined as

$$\text{supp}(A) = \{x \in X \mid \mu_A(x) > 0\}.$$

3. The *ceiling* of A is defined as

$$\text{ceil}(A) = \{x \in X \mid \mu_A(x) = \text{height}(A)\}.$$

4. The *kernel* of A is defined as

$$\text{kern}(A) = \{x \in X \mid \mu_A(x) = 1\}.$$

5. A is called *normal* if and only if there exists an element $x_0 \in X$ such that $\mu_A(x_0) = 1$ or, equivalently, if $\text{kern}(A) \neq \emptyset$.

In the following, we will denote the set of normal fuzzy subsets of X with $\mathcal{F}_N(X)$ and the set of fuzzy sets with a height of 1 with $\mathcal{F}_H(X)$. The set of all fuzzy subsets with non-empty ceiling will be denoted as $\mathcal{F}_T(X)$.

2.4. Definition. Consider a fuzzy set $A \in \mathcal{F}(X)$.

1. For a value $\alpha \in [0,1)$, the *strict α-cut* of A is defined as

$$[A]_{\underline{\alpha}} = \{x \in X \mid \mu_A(x) > \alpha\}.$$

2. For a value $\alpha \in [0,1]$, the *(non-strict) α-cut* of A is defined as

$$[A]_{\alpha} = \{x \in X \mid \mu_A(x) \geq \alpha\}.$$

2.5. Lemma. *For any fuzzy set $A \in \mathcal{F}(X)$, the following assertions hold:*

1. $\mathrm{supp}(A) = [A]_{\underline{0}}$ *and* $\mathrm{kern}(A) = [A]_1$

2. *Both kinds of α-cuts are nested sequences of sets:*

$$\begin{aligned} \forall \alpha, \beta \in [0,1): \quad \alpha \leq \beta \quad &\Longrightarrow \quad [A]_{\underline{\alpha}} \supseteq [A]_{\underline{\beta}} \\ \forall \alpha, \beta \in [0,1]: \quad \alpha \leq \beta \quad &\Longrightarrow \quad [A]_{\alpha} \supseteq [A]_{\beta} \end{aligned}$$

3. *Non-strict α-cuts are continuous in the following sense:*

$$[A]_{\alpha} = \bigcap_{\beta \in [0,\alpha)} [A]_{\beta}$$

4. *If the same is done for strict cuts a non-strict α-cut is obtained:*

$$[A]_{\alpha} = \bigcap_{\beta \in [0,\alpha)} [A]_{\underline{\beta}}$$

5. *Every fuzzy set A can be reconstructed from both kinds of its α-cuts, where* $\sup \emptyset$ *is defined to be* 0*:*

$$\mu_A(x) = \sup\{\alpha \mid x \in [A]_{\underline{\alpha}}\} = \sup\{\alpha \mid x \in [A]_{\alpha}\}$$

Proof. Assertions 1. and 2. follow trivially from the definition; 3. and 5. see, for instance, [40]. 4. can be proved similarly. □

It is not surprising that subsethood transfers to all α-cuts, strict and non-strict, and vice versa.

2.6. Lemma. *For any two fuzzy subsets $A, B \in \mathcal{F}(X)$, the following holds:*

$$A \subseteq B \iff \big(\forall \alpha \in [0,1]: \; [A]_{\alpha} \subseteq [B]_{\alpha}\big) \iff \big(\forall \alpha \in [0,1): \; [A]_{\underline{\alpha}} \subseteq [B]_{\underline{\alpha}}\big)$$

Proof. Trivial. □

2.1.2 Ordering-Based Convexity

The notion of convexity will be essential for Chapters 5 and 6. Throughout this subsection, let X be a non-empty set equipped with an ordering $\lesssim$.

2.7. Definition. A crisp subset M of X is called *connected* if and only if there is no sequence $x \lesssim y \lesssim z$, such that $x \in M$, $z \in M$, and $y \notin M$.

2.8. Definition. A fuzzy subset A of X is called *convex* if and only if, for every sequence $x \lesssim y \lesssim z$, the following inequality holds:

$$\mu_A(y) \geq \min(\mu_A(x), \mu_A(z))$$

As a first important result on convexity, we show that it directly corresponds to the connectedness of all α-cuts, regardless whether strict or non-strict.

2.9. Proposition. *The following three statements are equivalent for any fuzzy subset A of X:*

(i) A is convex.

(ii) Every strict α-cut is connected.

(iii) Every α-cut is connected.

Proof. (i)⇒(ii): Assume that there is an $\alpha \in [0, 1)$ such that $[A]_{\underline{\alpha}}$ is not connected, i.e. there is a sequence $x \lesssim y \lesssim z$ such that

$$\begin{aligned} x \in [A]_{\underline{\alpha}} &\implies \mu_A(x) > \alpha \\ z \in [A]_{\underline{\alpha}} &\implies \mu_A(z) > \alpha \\ y \notin [A]_{\underline{\alpha}} &\implies \mu_A(y) \leq \alpha \end{aligned}$$

which implies the contradiction $\mu_A(y) < \min(\mu_A(x), \mu_A(z))$.

(i)⇒(iii): Analogous to (i)⇒(ii).

(ii)⇒(i): Let $x \lesssim y \lesssim z$ be an ascending sequence. In the case

$$\min(\mu_A(x), \mu_A(z)) = 0,$$

nothing is to prove. So, assume the opposite and we have, for every $\alpha \in [0, 1)$ fulfilling

$$\alpha < \min(\mu_A(x), \mu_A(z))$$

that $x \in [A]_{\underline{\alpha}}$ and $z \in [A]_{\underline{\alpha}}$. Since $[A]_{\underline{\alpha}}$ is connected, y must be in $[A]_{\underline{\alpha}}$ as well:

$$\forall \alpha < \min(\mu_A(x), \mu_A(z)) : \ \mu_A(y) > \alpha.$$

Of course, this implies that $\mu_A(y) \geq \min(\mu_A(x), \mu_A(z))$.

(iii)⇒(i): Again, let $x \lesssim y \lesssim z$ be an ascending sequence. With the setting

$$\beta = \min(\mu_A(x), \mu_A(z)),$$

we obtain that x and z are both elements of $[A]_\beta$. Since $[A]_\beta$ is connected, it must contain y, too. Hence, the following must hold:

$$\mu_A(y) \geq \beta = \min(\mu_A(x), \mu_A(z))$$ □

On the other hand, it is worth to mention that there is no one-to-one correspondence between convexity of fuzzy sets and the convexity or concavity of its membership function in analytical terms. As follows next, monotonicity is already a sufficient condition for convexity, while this is, certainly, not true when it concerns convexity or concavity of functions in the usual sense.

2.10. Proposition. *A fuzzy set, the membership function of which is either non-decreasing or non-increasing, is convex.*

Proof. Without loss of generality, assume that some fuzzy set A has a non-decreasing membership function, i.e.

$$\forall x, y \in X : \ x \lesssim y \Longrightarrow \mu_A(x) \leq \mu_A(y).$$

Now take an arbitrary sequence $x \lesssim y \lesssim z$. Then non-decreasingness entails

$$\mu_A(x) \leq \mu_A(y) \leq \mu_A(z).$$

That implies, of course,

$$\mu_A(y) \geq \mu_A(x) = \min(\mu_A(x), \mu_A(z)).$$

The same argument can be applied to prove the corresponding assertion for non-increasingness. □

Finally, provided that $\lesssim$ is linear, we can show how convexity interacts with monotonicity.

2.11. Proposition. *Suppose that the ordering $\lesssim$ is linear. Then a fuzzy set $A \in \mathcal{F}(X)$ is convex if and only if there exists a partition of X into two connected subsets X_1, X_2, such that μ_A is non-decreasing over X_1 and non-increasing over X_2 and such that X_1 is completely below X_2:*

$$\forall x_1 \in X_1 \; \forall x_2 \in X_2 : \quad x_1 \lesssim x_2 \tag{2.1}$$

Proof. Assume that A is convex. For all $\alpha < \text{height}(A)$, we define X_1^α as the set of all elements left of $[A]_\alpha$ and X_2^α to contain all the elements right of $[A]_\alpha$:

$$\begin{aligned} X_1^\alpha &= \{x \mid \mu_A(x) < \alpha \wedge \forall y \in [A]_\alpha : \; y \gtrsim x\} \\ X_2^\alpha &= \{x \mid \mu_A(x) < \alpha \wedge \forall y \in [A]_\alpha : \; y \lesssim x\} \end{aligned}$$

Of course, the above set systems are nested in the sense

$$\begin{aligned} \forall \alpha, \beta \in [0, \text{height}(A)) : \quad \alpha \le \beta \quad &\Longrightarrow \quad X_1^\alpha \subseteq X_1^\beta, \\ \forall \alpha, \beta \in [0, \text{height}(A)) : \quad \alpha \le \beta \quad &\Longrightarrow \quad X_2^\alpha \subseteq X_2^\beta. \end{aligned}$$

Since $\lesssim$ is a linear ordering and since all $[A]_\alpha$ are connected (cf. Proposition 2.9), one easily verifies that each triplet $(X_1^\alpha, X_2^\alpha, [A]_\alpha)$ forms a partition of X such that the α-cut separates X_1^α and X_2^α:

$$\forall x_1 \in X_1^\alpha \; \forall \bar{x} \in [A]_\alpha \; \forall x_2 \in X_2^\alpha : \; x_1 \lesssim \bar{x} \lesssim x_2$$

Now let us prove that μ_A is non-decreasing on any X_1^α and non-increasing on any X_2^α. If there are two elements $x \lesssim y$ in X_1^α for which $\mu_A(x) > \mu_A(y)$, then we can choose any $z \in [A]_\alpha$ such that $x \lesssim y \lesssim z$, but

$$\min(\mu_A(x), \mu_A(z)) = \mu_A(x) > \mu_A(y),$$

and we obtain a contradiction. The same argument can be applied analogously to prove non-increasingness over X_2^α.

Now let us define

$$\begin{aligned} \tilde{X}_1 &= \bigcup_{\alpha \in [0, \text{height}(A))} X_1^\alpha, \\ \tilde{X}_2 &= \bigcup_{\alpha \in [0, \text{height}(A))} X_2^\alpha. \end{aligned}$$

Obviously, $\tilde{X}_1$, $\tilde{X}_2$, and $[A]_{\text{height}(A)}$ are disjoint and separated:

$$\begin{array}{lll} \forall x \in \tilde{X}_1 & \forall y \in \tilde{X}_2 : & x \lesssim y \\ \forall x \in \tilde{X}_1 & \forall y \in [A]_{\text{height}(A)} : & x \lesssim y \\ \forall x \in [A]_{\text{height}(A)} & \forall y \in X_2 : & x \lesssim y \end{array}$$

From Lemma 2.5, Point 3., and the de Morgan laws for crisp sets, we can deduce that these three sets, indeed, form a partition of X. Furthermore, μ_A is non-decreasing over $\tilde{X}_1$ and non-increasing over $\tilde{X}_2$. Finally, we can define

$$\begin{aligned} X_1 &= \tilde{X}_1 \cup [A]_{\text{height}(A)}, \\ X_2 &= \tilde{X}_2, \end{aligned}$$

which completes the construction. Of course, the monotonicity over $\tilde{X}_1$ cannot be deteriorated by the union with $[A]_{\text{height}(A)}$.

Reversely, suppose that there is a partition (X_1, X_2) fulfilling the above properties. Every sequence $x \lesssim y \lesssim z$ can be assigned to one of the following four cases:

1. $x \in X_1, z \in X_1$: Since X_1 is connected, it must also contain y and we obtain

   $$\mu_A(y) \geq \mu_A(x) = \min(\mu_A(x), \mu_A(z)).$$

2. $x \in X_2, z \in X_2$: Analogous to 1.

3. $x \in X_2, z \in X_1$: Impossible case since it contradicts to (2.1).

4. $x \in X_1, z \in X_2$: Then there are two possible cases

   $$\begin{array}{rcl} y \in X_1 & \Longrightarrow & \mu_A(y) \geq \mu_A(x) \\ y \in X_2 & \Longrightarrow & \mu_A(y) \geq \mu_A(z) \end{array}$$

   which, together, imply $\mu_A(y) \geq \min(\mu_A(x), \mu_A(z))$. □

The above ordering-based definition was chosen in order to be able to express convexity for arbitrary, even partially ordered domains. The usual definition, as already introduced by Zadeh in 1965 [72], however, is only applicable if X is a linear vector space over the reals (a Euclidean space $\mathbb{R}^n$ in the simplest case), where a fuzzy set A is called convex if all its α-cuts are convex subsets of X. Equivalently [44],

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$$

for all $x_1, x_2 \in X$ and all $\lambda \in [0,1]$. Obviously, this concept is also applicable if there is no canonical linear ordering, i.e. if the dimension of X is greater that 1. In the case $X = \mathbb{R}$, the two definitions coincide, due to the obvious equivalence

$$\forall x, y, z \in \mathbb{R}: \; x < y < z \iff \big(\exists \lambda \in (0,1): \; y = \lambda x + (1-\lambda)z\big).$$

Unless stated otherwise, we will use the ordering-based concept of convexity.

2.2 Logical Operations

Operations on crisp sets, such as intersection, union, complement, difference, etc., rely on logical operators, such as conjunction, disjunction, or negation. Membership degrees can be regarded as truth values; therefore, it is sufficient to define generalizations of logical operators for the domain $[0, 1]$ in order to define fuzzy intersections, unions, complements, and so forth.

2.2.1 Triangular Norms and Conorms

Closely related to Gödels intuitionistic logic [22], Zadeh suggested the minimum as fuzzy conjunction [72]. Although not an explicit element of Łukasiewicz logic, it is easily possible to extract

$$\max(x + y - 1, 0)$$

as the underlying conjunction. In probability theory, which is considered to be a field at least related to fuzzy set theory[1], the product is the common "conjunction".

At the beginning of the 1980s, triangular norms were discovered as a unifying framework for fuzzy conjunctions [36, 70]. Originally, they were used to formulate triangle inequalities in probabilistic metric spaces [60, 61].

2.12. Definition. A function $T : [0, 1]^2 \to [0, 1]$ is called *triangular norm (t-norm)* if and only if it fulfills the following properties for all $x, y, z \in [0, 1]$:

$$\begin{array}{rll}
(i) & T(x, 1) = x & \text{(neutral element)} \\
(ii) & x \leq y \Longrightarrow T(x, z) \leq T(y, z) & \text{(monotonicity)} \\
(iii) & T(x, y) = T(y, x) & \text{(commutativity)} \\
(iv) & T(x, T(y, z)) = T(T(x, y), z) & \text{(associativity)}
\end{array}$$

It is obvious that, for a t-norm T, the equalities $T(x, 0) = T(0, x) = 0$ hold. Commutativity implies that it must be non-decreasing in both arguments.

Monotonicity directly implies a generalization to arbitrary sequences as stated in the following lemma.

[1]After a long series of discussions, it is now a well-accepted fact that there is no one-to-one correspondence between fuzziness and probabilistic uncertainty concerning both semantics and truth functionality [28].

2.13. Lemma. *For a t-norm T, a sequence $(x_i)_{i\in I}$, and a value y in the unit interval, the following inequalities hold:*

$$T(\inf_{i\in I} x_i, y) \leq \inf_{i\in I} T(x_i, y)$$
$$T(\sup_{i\in I} x_i, y) \geq \sup_{i\in I} T(x_i, y)$$

Proof. Follows directly from the monotonicity of t-norms. □

In fact, all the above mentioned operations are t-norms:

$$T_{\mathbf{M}}(x, y) = \min(x, y)$$
$$T_{\mathbf{P}}(x, y) = x \cdot y$$
$$T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0)$$

Consequently, we will speak of the minimum t-norm, the product t-norm, and the Łukasiewicz t-norm.

2.14. Definition. Of course, we can restrict the well-known partial ordering of real-valued functions such that a partial ordering of t-norms is obtained:

$$T_1 \leq T_2 \iff \forall x, y \in [0, 1] : \ T_1(x, y) \leq T_2(x, y).$$

If $T_1 \leq T_2$, we will say that T_1 is *weaker* than T_2 or, equivalently, that T_2 is *stronger* than T_1.

It is easy to see from the monotonicity and neutrality of 1, that $T_{\mathbf{M}}$ is the strongest t-norm with respect to the above ordering and that the so-called drastic product

$$T_{\mathbf{W}}(x, y) = \begin{cases} \min(x, y) & \text{if } \max(x, y) = 1 \\ 0 & \text{otherwise} \end{cases}$$

is the weakest t-norm. Taking into account that, on the unit square, the inequality $x + y - 1 \leq xy$ holds, we obtain

$$T_{\mathbf{W}} < T_{\mathbf{L}} < T_{\mathbf{P}} < T_{\mathbf{M}}.$$

There are a lot of other t-norms and parametrized families of t-norms [37, 40]. Among them, there is one outstanding class—the family of Frank t-norms $(T^{\mathbf{F}}_\lambda)_{\lambda\in[0,\infty]}$—which provides a continuous and monotonic transition between three of our basic t-norms:

$$T^{\mathbf{F}}_\lambda(x, y) = \begin{cases} T_{\mathbf{M}}(x, y) & \text{if } \lambda = 0 \\ T_{\mathbf{P}}(x, y) & \text{if } \lambda = 1 \\ T_{\mathbf{L}}(x, y) & \text{if } \lambda = \infty \\ \log_\lambda\left(1 + \frac{(\lambda^x - 1)(\lambda^y - 1)}{\lambda - 1}\right) & \text{otherwise} \end{cases}$$

2.15. Definition. For an $x \in [0,1]$ and an $n \in \mathbb{N}$, we define the n-th power with respect to a t-norm T as

$$x_T^{(n)} = \begin{cases} x & \text{if } n = 1, \\ T(\overbrace{x, \ldots, x}^{n \text{ times}}) & \text{otherwise.} \end{cases}$$

2.16. Definition. Some basic properties of t-norms:

1. A t-norm T is called *strictly monotone* if and only if

$$\forall x \in (0,1]\ \forall y, z \in [0,1] : \ y < z \Longrightarrow T(x,y) < T(x,z).$$

2. A strictly monotone and continuous t-norm is called *strict*.

3. A t-norm T is called left-continuous if, for each $x \in [0,1]$, the component mapping $T(x, .)$ is left-continuous.

4. A t-norm T is called *Archimedean* if and only if, for all pairs $(x,y) \in (0,1)^2$, there is an $n \in \mathbb{N}$ such that

$$x_T^{(n)} < y.$$

5. A t-norm T is called *nilpotent* if it is continuous and, for each $x \in (0,1)$, there exists an $n \in \mathbb{N}$ such that

$$x_T^{(n)} = 0.$$

For a detailed investigation of the connections between the different properties we refer to [37]. We shall restrict ourselves to some fundamental results which will be important for our further studies.

2.17. Theorem. *A function $T : [0,1]^2 \to [0,1]$ is a continuous Archimedean t-norm if and only if there exists a continuous, strictly decreasing function $f : [0,1] \to [0,\infty]$ with $f(1) = 0$ called* additive generator *such that, for all $x, y \in [0,1]$, the following holds:*

$$T(x,y) = f^{-1}\big(\min(f(x) + f(y), f(0))\big)$$

The generator f is uniquely determined up to a positive multiplicative constant.

Proof. See [42, 60, 61]. □

2.18. Definition. Two t-norms T_1 and T_2 are called *isomorphic* if and only if there exists an automorphism, i.e. a strictly increasing bijection, $\varphi : [0,1] \to [0,1]$, such that, for all pairs (x, y) in the unit square,

$$T_2(x, y) = \varphi^{-1}\big(T_1(\varphi(x), \varphi(y))\big).$$

2.19. Theorem. *Provided that $T : [0,1]^2 \to [0,1]$ is a binary operation on the unit interval, the following holds:*

1. *T is a nilpotent t-norm if and only if it is isomorphic to the Łukasiewicz t-norm.*

2. *T is a strict t-norm if and only if it is isomorphic to the product t-norm.*

Proof. See [42, 52, 61]. □

In an analogous way, a unifying concept for studying fuzzy disjunctions has been proposed.

2.20. Definition. A function $S : [0,1]^2 \to [0,1]$ is called *triangular conorm (t-conorm)* if and only if it fulfills the following properties for all x, y, z in the unit interval:

(i)	$S(x, 0) = x$	(neutral element)
(ii)	$x \leq y \Longrightarrow S(x, z) \leq S(y, z)$	(monotonicity)
(iii)	$S(x, y) = S(y, x)$	(commutativity)
(iv)	$S(x, S(y, z)) = S(S(x, y), z)$	(associativity)

Obviously, the only difference between t-norms and t-conorms is that 0 is the neutral element instead of 1 and we can deduce that $S(x, 1) = 1$ for all $x \in [0,1]$. Again, commutativity implies that S must be non-decreasing in both components.

It is easy to prove that, for any t-norm T, the mapping defined as

$$S(x, y) = 1 - T(1 - x, 1 - y) \tag{2.2}$$

is a t-conorm. Reversely, every t-conorm S can be constructed by the above formula from its dual t-norm given as

$$T(x, y) = 1 - S(1 - x, 1 - y). \tag{2.3}$$

If this construction principle is applied to our four basic t-norms, we obtain the following:

$$\begin{aligned} S_{\mathbf{M}}(x,y) &= \max(x,y) \\ S_{\mathbf{P}}(x,y) &= x+y-x\cdot y \\ S_{\mathbf{L}}(x,y) &= \min(x+y,1) \\ S_{\mathbf{W}}(x,y) &= \begin{cases} \max(x,y) & \text{if } \min(x,y)=0 \\ 1 & \text{otherwise} \end{cases} \end{aligned}$$

In fact, the maximum was already suggested by Zadeh as disjunction. The mapping $S_{\mathbf{P}}$ is known as *algebraic sum* which is well-known in probability theory. Again, $S_{\mathbf{L}}$ is an operation which can be defined in the framework of Łukasiewicz logic; sometimes it is called *bounded sum*.

Analogously, one can prove that $S_{\mathbf{M}}$ is the weakest and $S_{\mathbf{W}}$—the so-called drastic sum—is the strongest t-conorm and that the following holds:

$$S_{\mathbf{M}} < S_{\mathbf{P}} < S_{\mathbf{L}} < S_{\mathbf{W}}$$

By applying (2.2) to the family of Frank t-norms we obtain the family of Frank t-conorms $(S^{\mathbf{F}}_{\lambda})_{\lambda\in[0,\infty]}$ which, analogously, unifies the three basic t-conorms $S_{\mathbf{M}}$, $S_{\mathbf{P}}$, and $S_{\mathbf{L}}$:

$$S^{\mathbf{F}}_{\lambda}(x,y) = \begin{cases} S_{\mathbf{M}}(x,y) & \text{if } \lambda=0 \\ S_{\mathbf{P}}(x,y) & \text{if } \lambda=1 \\ S_{\mathbf{L}}(x,y) & \text{if } \lambda=\infty \\ 1-\log_{\lambda}\left(1+\frac{(\lambda^{1-x}-1)(\lambda^{1-y}-1)}{\lambda-1}\right) & \text{otherwise} \end{cases}$$

It is worth to mention that $\left((T^{\mathbf{F}}_{\lambda}, S^{\mathbf{F}}_{\lambda})\right)_{\lambda\in[0,\infty]}$ are the only couples of t-norms and t-conorms, which are dual to each other in the sense of (2.2) and (2.3), such that the following functional equation is fulfilled for all pairs (x,y) in the unit square [20]:

$$T(x,y)+S(x,y)=x+y$$

All definitions of properties and representations can be carried over to t-conorms in a dual way. We omit to present that in detail here.

Finally, we can define the intersection and union of two fuzzy subsets.

2.21. Definition. Let A, B be two fuzzy subsets of the same domain X. Then their *fuzzy intersection* with respect to a t-norm T, short *T-intersection*, denoted as $A\cap_T B$, is represented by the membership function

$$\mu_{A\cap_T B}(x) = T(\mu_A(x), \mu_B(x)).$$

The *fuzzy union* of A and B with respect to a given t-conorm S, short *S-union*, denoted $A \cup_S B$, is represented by the membership function

$$\mu_{A\cup_S B}(x) = S(\mu_A(x), \mu_B(x)).$$

2.2.2 Negations

The formulae (2.2) and (2.3)—well-known as de Morgan laws for t-norms and t-conorms—implicitly use

$$N_{\mathbf{S}}(x) = 1 - x \tag{2.4}$$

as negation. As a matter of fact, it appears to be the most important fuzzy negation in fuzzy systems applications. Note that this operation appears in Zadeh's definition of the complement of a fuzzy set, in Łukasiewicz logic, and in probability theory for computing the probability of the complement event. Thus, it is justified to call it *standard negation*. In intuitionistic logic, however,

$$N_{\mathbf{I}}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

is used as negation.

Similar to conjunctions and disjunctions, a generalized framework for studying negations has been introduced.

2.22. Definition. A non-increasing function $N : [0,1] \to [0,1]$ fulfilling the boundary conditions $N(0) = 1$ and $N(1) = 0$ is called *negation*.

2.23. Definition. A negation is called *strict* if and only if it is strictly decreasing and continuous. A strict negation N is called *involution* if and only if it is self-inverse, i.e.

$$\forall x \in [0,1] : \ N(N(x)) = x.$$

First of all, it is trivial to check that $N_{\mathbf{I}}$ is the smallest and that the so-called dual intuitionistic negation

$$N_{\mathbf{D}}(x) = \begin{cases} 0 & \text{if } x = 1 \\ 1 & \text{otherwise} \end{cases}$$

is the greatest negation, where none of them is neither an involution nor strict. The standard negation is, of course, an involution. Now let us see in which way strict negations and involutions are represented as transformations of the standard negation.

2.24. Theorem. *For a negation N, the following two equivalences hold:*

1. *N is an involution if and only if there exists an automorphism φ of the unit interval such that N can be represented as follows:*

$$\forall x \in [0,1]: \ N(x) = \varphi^{-1}(1 - \varphi(x))$$

2. *N is strict if and only if there exist two automorphisms φ, ψ of the unit interval such that N can be represented as follows:*

$$\forall x \in [0,1]: \ N(x) = \psi(1 - \varphi(x))$$

Proof. 1. see [57]; 2. see [17]. □

The starting points of this investigation of generalized negations were the de Morgan laws for t-norms and t-conorms. If we replace $N_{\mathbf{S}}$ by arbitrary negations N_1, N_2 we obtain

$$\begin{aligned} S(x,y) &= N_1^{-1}\big(T(N_1(x), N_1(x))\big) \\ T(x,y) &= N_2^{-1}\big(T(N_2(x), N_2(x))\big) \end{aligned}$$

as de Morgan laws, where T denotes an arbitrary t-norm and S denotes a t-conorm. Obviously, this definition only makes sense if N_1 and N_2 are strict, otherwise the inverse functions would not be definable.

2.25. Proposition. *Assume that N is a strict negation.*

1. *For any t-norm T, the following mapping defines a t-conorm:*

$$S(x,y) = N^{-1}(T(N(x), N(y)))$$

2. *For any t-conorm S, the following mapping defines a t-norm:*

$$T(x,y) = N^{-1}(S(N(x), N(y)))$$

Proof. See [70]. □

Moreover, it is easy to see that, applying the two de Morgan laws successively, only yields the original operation if the negation is an involution.

In [2, 18], even a necessary and sufficient condition is provided, under which it is possible to define a negation such that the de Morgan laws hold for a given Archimedean t-norm and an Archimedean t-conorm.

The de Morgan laws represent two such important properties that an own name was introduced for combinations of t-norms, t-conorms, and negations satisfying them.

2.26. Definition. A triple (T, S, N), where T is a t-norm, S is a t-conorm, and N is a strict negation, is called *de Morgan triple* if and only if the first de Morgan law

$$S(x, y) = N^{-1}\big(T(N(x), N(y))\big)$$

is fulfilled for all $x, y \in [0, 1]$. A de Morgan triple is called *continuous* if T and S are continuous. A continuous de Morgan triple (T, S, N) is called *Łukasiewicz triple* in the case that N is an involution and T is nilpotent.

Finally, in accordance to our original intention, we can define the complement of a fuzzy set.

2.27. Definition. Let A be a fuzzy subset of a domain X. Then its *fuzzy complement* with respect to a negation N, short *N-complement*, denoted as $\complement_N A$, is represented by the membership function

$$\mu_{\complement_N A}(x) = N(\mu_A(x)).$$

2.2.3 Implications

The last remaining important connective is implication. In classical Boolean logic, the implication can be defined from negation and disjunction as

$$\neg x \vee y.$$

If (T, S, N) is a de Morgan triple, the same can be done in the fuzzy case:

$$I_T(x, y) = S(N(x), y)$$

This type of implication is called *S-implication*. Besides, there is a vast number of equivalent ways in Boolean logic to define the implication from the basic operations $\wedge$, $\vee$, and $\neg$. In the fuzzy case, these definitions are not necessarily equivalent. Unfortunately, S-implications and other definitions suffer from bad logical properties, e.g. non-transitivity (see later). An alternative definition which overcomes these difficulties is the so-called residuum.

2.28. Definition. Let T be a t-norm. A function $R : [0, 1]^2 \to [0, 1]$ is called *residual implication (residuum)* of T if and only if the following equivalence is fulfilled for all $x, y, z \in [0, 1]$:

$$T(x, y) \leq z \iff x \leq R(y, z)$$

2.29. Lemma. *Assume that T is an arbitrary t-norm for which a residuum R exists. Then $R(1,z)=z$ holds for all $z \in [0,1]$.*

Proof. Assigning 1 to y in the definition of the residuum, we obtain

$$x = T(x,1) \leq z \iff x \leq R(1,z),$$

which directly implies $R(1,z)=z$ (see also [49]). □

2.30. Lemma. *For any left-continuous t-norm T, there exists a unique residuum $\vec{T}$ which is given as*

$$\vec{T}(x,y) = \sup\{u \in [0,1] \mid T(u,x) \leq y\}. \tag{2.5}$$

Proof. See [16, 40]. □

Furthermore, there is an axiomatic approach to fuzzy implications, which was originally introduced by W. Pedrycz [59] and, later on, studied intensively by S. Gottwald [24, 26].

2.31. Definition. Assume that T is a t-norm. A function $\varphi : [0,1]^2 \to [0,1]$ is called Φ-operator of T if and only if the following axioms are satisfied for all $x,y,z \in [0,1]$:

(i)	$y \leq z \Longrightarrow \varphi(x,y) \leq \varphi(x,z)$	(monotonicity in second component)
(ii)	$T(x,\varphi(x,y)) \leq y$	(modus ponens)
(iii)	$y \leq \varphi(x,T(x,y))$	(exchange rule)

It has been shown [24] that a unique Φ-operator exists for a left-continuous t-norm, in which case it coincides with the residuum $\vec{T}$. Hence, all the above properties are satisfied for residual implications of left-continuous t-norms. Moreover, there are some other relationships which are worth to be mentioned.

2.32. Lemma. *The following basic properties hold for the residuum $\vec{T}$ of any left-continuous t-norm T:*

1. $\forall x,y \in [0,1]: \ x \leq y \Longleftrightarrow \vec{T}(x,y) = 1$
2. $\forall x,y,z \in [0,1]: \ x \leq y \Longrightarrow \vec{T}(x,z) \geq \vec{T}(y,z)$
3. $\forall x,y,z \in [0,1]: \ y \leq z \Longrightarrow \vec{T}(x,y) \leq \vec{T}(x,z)$
4. $\forall x,y,z \in [0,1]: \ T\big(\vec{T}(x,y),\vec{T}(y,z)\big) \leq \vec{T}(x,z)$

5. $\forall x,y,z\in[0,1]:\ \vec{T}\big(T(x,y),z\big)\le\vec{T}\big(x,\vec{T}(y,z)\big)$

6. $\forall x,y\in[0,1]:\ T(x,\vec{T}(x,y))\le y$

7. $\forall x,y\in[0,1]:\ y\le\vec{T}(x,T(x,y))$

8. $\forall x,y,z\in[0,1]:\ \vec{T}(x,y)\le\vec{T}\big(T(x,z),T(y,z)\big)$

9. $\forall x,y,z\in[0,1]:\ \min\big(\vec{T}(x,y),\vec{T}(x,z)\big)=\vec{T}\big(x,\min(y,z)\big)$

10. $\forall x,y,z\in[0,1]:\ \min\big(\vec{T}(x,z),\vec{T}(y,z)\big)=\vec{T}\big(\max(x,y),z\big)$

Proof. 1.–8. see [26]. 9. follows directly from the fact that $\vec{T}$ is non-decreasing in the second component (cf. 3.) while, analogously, 10. is an immediate consequence of 2. □

Point 4., obviously, states that implications can be chained. This so-called transitivity of residual implications, as it will turn out later, is an extremely important property in various respects, especially when it concerns deduction.

2.33. Lemma. *For any left-continuous t-norm T, the residuum $\vec{T}$ is left-continuous in its first and right-continuous in its second argument.*

Proof. Left-continuity of the first argument means that, for all sequences $(x_i)_{i\in I}$ and values y in the unit interval, the following equality holds:

$$\vec{T}\big(\sup_{i\in I}x_i,y\big)=\inf_{i\in I}\vec{T}(x_i,y)$$

Due to the non-increasingness in the first argument, the inequality

$$\vec{T}\big(\sup_{i\in I}x_i,y\big)\le\inf_{i\in I}\vec{T}(x_i,y)\tag{2.6}$$

always holds automatically. The left-hand side is defined as

$$\sup\{u\in[0,1]\mid T(\sup_{i\in I}x_i,u)\le y\}.$$

Thus, it is sufficient for the equality in (2.6) to show that

$$T\big(\sup_{i\in I}x_i,\inf_{j\in I}\vec{T}(x_j,y)\big)\le y$$

holds. For this purpose, consider left-continuity of T:

$$\begin{aligned} T\bigl(\sup_{i\in I} x_i, \inf_{j\in I} \vec{T}(x_j,y)\bigr) &= \sup_{i\in I} T\bigl(x_i, \inf_{j\in I} \vec{T}(x_j,y)\bigr) \\ &\leq \sup_{i\in I} T\bigl(x_i, \vec{T}(x_i,y)\bigr) \end{aligned}$$

Finally the modus ponens rule (Lemma 2.32, Point 6.) yields

$$\sup_{i\in I} T\bigl(x_i, \vec{T}(x_i,y)\bigr) \leq \sup_{i\in I} y = y.$$

Right-continuity in the second argument can be shown applying similar arguments (see [49] for details). □

Now let us see what we obtain if we construct implications from the three left-continuous ones of our four basic t-norms, where we use the standard negation for constructing the S-implication:

$$I_{T_{\mathbf{M}}}(x,y) = \max(1-x,y) \qquad \vec{T}_{\mathbf{M}}(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

$$I_{T_{\mathbf{P}}}(x,y) = 1-x+x\cdot y \qquad \vec{T}_{\mathbf{P}}(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{otherwise} \end{cases}$$

$$I_{T_{\mathbf{L}}}(x,y) = \min(1-x+y,1) \qquad \vec{T}_{\mathbf{L}}(x,y) = \min(1-x+y,1)$$

Clearly, only for $T_{\mathbf{L}}$ the two types of implications coincide. For obvious reasons, $\vec{T}_{\mathbf{L}}$ is called Łukasiewicz implication. It is easy to check that $I_{T_{\mathbf{M}}}$ and $I_{T_{\mathbf{P}}}$ are not transitive in the sense of Lemma 2.32, Point 4. They are called Kleene-Dienes and Reichenbach implications, respectively. The two residua $\vec{T}_{\mathbf{M}}$ and $\vec{T}_{\mathbf{P}}$ are not continuous. Anyway, both fulfill all properties of Lemma 2.32 and Definition 2.31. They are called Gödel and Goguen implication.

It remains to clarify in which way a residuum of a t-norm can be represented by means of its additive generator.

2.34. Theorem. *If T is a continuous Archimedean t-norm with additive generator f, its residuum is given as*

$$\vec{T}(x,y) = f^{-1}\bigl(\max(f(y)-f(x),0)\bigr).$$

Proof. See [18]. □

Theorem 2.34 can be used to prove a characterization of continuity of residual implications.

2.35. Corollary. *A residuum $\vec{T}$ of a t-norm T is continuous if and only if T is isomorphic to the Łukasiewicz t-norm, i.e. if T is nilpotent.*

Proof. See [62]. □

In Boolean logic, the conjunction and the implication can be used to define an equivalence. The same can be done for t-norms and their residua.

2.36. Definition. For a left-continuous t-norm T, the *biimplication* $\overleftrightarrow{T}$ is defined in the following way:

$$\overleftrightarrow{T}(x,y) = T\big(\vec{T}(x,y), \vec{T}(y,x)\big) = \min\big(\vec{T}(x,y), \vec{T}(y,x)\big) \tag{2.7}$$

2.37. Lemma. *The following holds for the biimplication $\overleftrightarrow{T}$ of an arbitrary left-continuous t-norm T:*

1. *$\overleftrightarrow{T}$ is a commutative operation.*
2. $\forall x,y \in [0,1] :\ x = y \iff \overleftrightarrow{T}(x,y) = 1$
3. $\forall x,y \in [0,1] :\ \overleftrightarrow{T}(x,y) = \vec{T}(\max(x,y), \min(x,y))$
4. $\forall x,y,z \in [0,1] :\ T(\overleftrightarrow{T}(x,y), \overleftrightarrow{T}(y,z)) \le \overleftrightarrow{T}(x,z)$
5. $\forall x,y,u,v \in [0,1] :\ \min\big(\overleftrightarrow{T}(x,y), \overleftrightarrow{T}(u,v)\big) \le \overleftrightarrow{T}\big(\min(x,u), \min(y,v)\big)$
6. $\forall x,y,u,v \in [0,1] :\ \min\big(\overleftrightarrow{T}(x,y), \overleftrightarrow{T}(u,v)\big) \le \overleftrightarrow{T}\big(\max(x,u), \max(y,v)\big)$

Proof. 1. is trivial; 2. follows directly from Lemma 2.32 and the definition; 3. see [18, 40]. 4. follows from basic properties of t-norms and the transitivity of $\vec{T}$ (compare with Lemma 2.32). Simply by using the definition and the monotonicities of $\vec{T}$ (cf. Lemma 2.32), we obtain

$$\begin{aligned}
\min\big(\overleftrightarrow{T}(x,y), \overleftrightarrow{T}(u,v)\big) &= \min\big(\vec{T}(x,y), \vec{T}(y,x), \vec{T}(u,v), \vec{T}(v,u)\big)\\
&= \min\big(\vec{T}(x,y), \vec{T}(u,v), \vec{T}(y,x), \vec{T}(v,u)\big)\\
&\le \min\big(\vec{T}(\min(x,u),y), \vec{T}(\min(x,u),v),\\
&\qquad\quad \vec{T}(\min(y,v),x), \vec{T}(\min(y,v),u)\big)\\
&= \min\big(\vec{T}(\min(x,u),\min(y,v)),\\
&\qquad\quad \vec{T}(\min(y,v),\min(x,u))\big)\\
&= \overleftrightarrow{T}\big(\min(x,u),\min(y,v)\big).
\end{aligned}$$

Point 5. can be shown applying similar monotonicity arguments. □

Taking our three basic left-continuous t-norms, we obtain the following biimplications:

$$\begin{aligned}\overset{\leftrightarrow}{T}_{\mathbf{M}}(x,y) &= \begin{cases} 1 & \text{if } x = y \\ \min(x,y) & \text{otherwise} \end{cases} \\ \overset{\leftrightarrow}{T}_{\mathbf{P}}(x,y) &= \begin{cases} 1 & \text{if } x = y = 0 \\ \frac{\min(x,y)}{\max(x,y)} & \text{otherwise} \end{cases} \\ \overset{\leftrightarrow}{T}_{\mathbf{L}}(x,y) &= 1 - |x-y|\end{aligned}$$

2.2.4 A Short Remark on Fuzzy Logic

The reader may have observed that the popular term *"fuzzy logic"* has not yet appeared until now—a term which has been used excessively as general expression for a lot of topics related to fuzzy sets[2], even if completely apart from logic. In order to overcome this misconception, it is now common to distinguish between fuzzy logic in the narrow and in the broad sense[3]. While fuzzy logic in the broad sense still covers anything related to fuzziness, fuzzy logic in the narrow sense, nowadays, appears to be an important subbranch of many-valued logic. Let us mention briefly what the basic elements of fuzzy logic in the narrow sense are.

First of all, the unit interval $[0,1]$ equipped with a de Morgan triple (T,S,N) does not necessarily provide the properties for being regarded seriously as a kind of logic (consider, for instance, non-transitivity which inhibits sequential deduction). Residuation based on continuous t-norms, however, offers a way to define algebraic structures with "real logical properties". Moreover, by adding the unary operation

$$N_T(x) = \vec{T}(x,0),$$

which is easily proved to be a negation [26], one can define t-norm-based propositional logics. For $T_{\mathbf{M}}$, the so-called *Gödel logic* is obtained. Starting from $T_{\mathbf{P}}$ results in the so-called *product logic*, while taking $T_{\mathbf{L}}$ produces what is commonly called *Łukasiewicz logic*. Any of these three logics is sound, complete, and axiomatizable [27]. Reversely, the axioms of these logics can be used as defining properties of generalized algebraic structures for many-valued logics, such as Heyting algebras corresponding to Gödel logic, MV-algebras corresponding to Łukasiewicz logic, and product algebras based on the axioms of product logic.

[2]Note that the expression "fuzzy logic" was established a considerable time after Zadeh's introduction of fuzzy sets.

[3]In fact, it was Zadeh himself who first claimed this distinction.

Moreover, by introducing the infimum as universal and the supremum as existential quantifier, it is possible to construct even predicate calculi based on t-norms. Only for Gödel logic, however, a recursively axiomatizable calculus is obtained, but neither for Łukasiewicz nor product logic [27].

2.3 The Extension Principle

While the purpose of the previous section was to provide fuzzifications of set operations, we shall now discuss how to extend ordinary crisp-to-crisp mappings such that they can be applied even to fuzzy sets. In the two-valued case, the image of a set $M \subseteq X$ under a function $f : X \to Y$ is usually defined as

$$f(M) = \{y \in Y \mid \exists x \in M : \; y = f(x)\}. \tag{2.8}$$

If we consider an arbitrary element $y \in X$, (2.8) immediately implies

$$y \in f(M) \iff \big(\exists x \in M : \; y = f(x)\big)$$

which can be translated to the fuzzy case directly by replacing the existential quantifier with the supremum. This idea is known as the so-called *extension principle* and goes back to Zadeh [76, 77, 78].

2.38. Definition. For a function $\phi : X \to Y$, the *extension* to fuzzy subsets is defined as an operator $\hat{\phi} : \mathcal{F}(X) \to \mathcal{F}(Y)$, where the image of a fuzzy subset $A \in \mathcal{F}(X)$ is represented by the membership function (with the convention $\sup \emptyset = 0$)

$$\mu_{\hat{\phi}(A)}(x) = \sup\{\mu_A(u) \mid x = \phi(u)\}. \tag{2.9}$$

The next theorem shows that the extension is nothing else than applying the function to each strict α-cut in the sense of (2.8).

2.39. Lemma. *For a function $\phi : X \to Y$ and a fuzzy subset $A \in \mathcal{F}(X)$ the following holds:*

$$\forall \alpha \in [0, 1) : \; [\hat{\phi}(A)]_{\underline{\alpha}} = \phi\big([A]_{\underline{\alpha}}\big)$$

Proof. Let ϕ, A, and α be arbitrary but fixed:

$$\begin{aligned} [\hat{\phi}(A)]_{\underline{\alpha}} &= \{y \in Y \mid \sup\{\mu_A(x) \mid y = \phi(x)\} > \alpha\} \\ &= \{y \in Y \mid \exists x \in X : \; y = \phi(x) \wedge \mu_A(x) > \alpha\} \\ &= \{y \in Y \mid \exists x \in [A]_{\underline{\alpha}} : \; y = \phi(x)\} \\ &= \phi\big([A]_{\underline{\alpha}}\big) \end{aligned}$$

□

Sometimes, the term extension principle is used generally whenever functions, properties, etc., are generalized to fuzzy sets just by applying them to each (strict) α-cut. Lemma 2.39 gives a motivation for this alternative understanding of the extension principle.

If we are considering an n-ary function of the type

$$\psi : X_1 \times \cdots \times X_n \longrightarrow Y,$$

the extension (2.9) can be applied to any fuzzy subset of the product space $X_1 \times \cdots \times X_n$ without any modification. If, however, only fuzzy subsets of the single components $A_i \in \mathcal{F}(X_i)$ are known, the computation of the extension, first of all, requires a way for combining them to a kind of fuzzy Cartesian product.

2.40. Definition. For fuzzy subsets A and B of domains X and Y, respectively, the *fuzzy Cartesian product* of A and B with respect to a t-norm T is represented by the following membership function:

$$\begin{array}{rcl} \mu_{A\times_T B} : \quad X \times Y & \longrightarrow & [0,1] \\ \mu_{A\times_T B}(x,y) & \longmapsto & T(\mu_A(x), \mu_B(y)) \end{array}$$

Since t-norms are associative, this construction can be generalized inductively to the n-ary case in a straightforward way.

2.41. Definition. Consider a Cartesian product $X = X_1 \times \cdots \times X_n$, a t-norm T, and a mapping $\psi : X \to Y$. For fuzzy subsets $A_1, \ldots, A_n$ of $X_1, \ldots, X_n$, respectively, the *T-extension* $\hat{\psi}(A_1, \ldots, A_n)$ is represented by the membership function

$$\mu_{\hat{\psi}(A_1,\ldots,A_n)}(y) = \sup\{T(\mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_n)) \mid y = \psi(x_1, \ldots, x_n)\}.$$

Obviously, this means nothing else than just the application of the extension (2.9) to the Cartesian product $A_1 \times_T \cdots \times_T A_n$. Hence, Lemma 2.39 entails

$$[\hat{\psi}(A_1, \ldots, A_n)]_{\underline{\alpha}} = \psi\big([A_1 \times_T \cdots \times_T A_n]_{\underline{\alpha}}\big).$$

As it is easily verified that

$$[A_1 \times_{T_\mathbf{M}} \cdots \times_{T_\mathbf{M}} A_n]_{\underline{\alpha}} = [A_1]_{\underline{\alpha}} \times \cdots \times [A_n]_{\underline{\alpha}},$$

we obtain the following result.

2.42. Corollary. *The $T_\mathbf{M}$-extension $\hat{\psi}$ of an arbitrary n-ary mapping ψ admits the following resolution:*

$$\forall \alpha \in [0,1): \ [\hat{\psi}(A_1, \ldots, A_n)]_{\underline{\alpha}} = \psi\big([A_1]_{\underline{\alpha}} \times \cdots \times [A_n]_{\underline{\alpha}}\big)$$

2.4 Binary Fuzzy Relations

2.4.1 Basic Notions and Properties

2.43. Definition. A function $R : X \times Y \to [0,1]$ is called *(binary) fuzzy relation* of type (X, Y). The value $R(x, y)$ is interpreted as the degree to which an $x \in X$ and a $y \in Y$ are in relation. If $X = Y$ we say that R is a fuzzy relation on X.

Of course, a fuzzy relation R is nothing else than a fuzzy subset of $X \times Y$. Therefore, the distinction between R and its membership function μ_R seems to be appropriate. Nevertheless, thinking of a fuzzy relation rather as a two-placed fuzzy predicate than as a fuzzy subset, we will simply omit this distinction.

2.44. Definition. Let N be a negation. If R is a fuzzy relation on a domain X, its *inverse* R^{-1} and its *dual* R^d are defined as

$$\begin{aligned} R^{-1}(x,y) &= R(y,x), \\ R^d(x,y) &= N(R(y,x)). \end{aligned}$$

While it is obvious how to define intersections, unions, and complements of fuzzy relations (see Definition 2.21 and 2.27), an important operation is the composition with respect to a given t-norm.

2.45. Definition. Assume that R is a fuzzy relation of type (X, Y) and Q is a fuzzy relation of type (Y, Z). Then the T*-composition* $R \circ_T Q$, where T denotes a t-norm, is defined as follows:

$$\begin{array}{rccl} R \circ_T Q : & X \times Z & \longrightarrow & [0,1] \\ & (x,z) & \longmapsto & \sup\limits_{y \in Y} T\big(R(x,y), Q(y,z)\big) \end{array}$$

It is worth to mention that the computation of the T-composition is monotonic in both components with respect to inclusion:

$$\begin{aligned} R_1 \subseteq R_2 &\implies R_1 \circ_T Q \subseteq R_2 \circ_T Q \\ Q_1 \subseteq Q_2 &\implies R \circ_T Q_1 \subseteq R \circ_T Q_2 \end{aligned}$$

Moreover, provided that T is left-continuous, one can prove that it is an associative operation.

As in the crisp case, there are some outstanding properties of fuzzy relations which are of special importance.

2.46. Definition. Consider a fuzzy relation R on a domain X, where T denotes a t-norm and S denotes a t-conorm:

1. R is called *reflexive* if and only if
$$\forall x \in X: \; R(x,x) = 1.$$
2. R is called *irreflexive* if and only if
$$\forall x \in X: \; R(x,x) = 0.$$
3. R is called *symmetric* if and only if
$$\forall x, y \in X: \; R(x,y) = R(y,x).$$
4. R is called T*-asymmetric* if and only if
$$\forall x, y \in X: \; T(R(x,y), R(y,x)) = 0.$$
5. R is called T*-antisymmetric* if and only if
$$\forall x, y \in X: \; x \neq y \implies T(R(x,y), R(y,x)) = 0.$$
6. R is called T*-transitive* if and only if
$$\forall x, y, z \in X: \; T(R(x,y), R(y,z)) \leq R(x,z).$$
7. R is called *negatively* S*-transitive* if and only if
$$\forall x, y, z \in X: \; R(x,z) \leq S(R(x,y), R(y,z)).$$
8. R is called T*-*S*-Ferrers* if and only if
$$\forall x, y, u, v \in X: \; T(R(x,y), R(u,v)) \leq S(R(x,v), R(u,y)).$$

Note that, in the framework of fuzzy propositional logic (see 2.2.4), $x \leq y$ can be interpreted as "$x \to y$ is a tautology" (see Lemma 2.32). Consequently, the meaning of T-transitivity is that the following is always true:

"x and y are R-related" and "y and z are R-related"
implies that
"x and z are R-related"

There are numerous results concerning the representation of T-transitive fuzzy relations. Since they are of no practical relevance for our further studies, the reader is referred to literature [18, 53, 64].

Another way of "ranking" t-norms is the concept of dominance. As we will see immediately, it directly corresponds to the preservation of T-transitivity.

2.47. Definition. A t-norm T_1 is said to *dominate* another t-norm T_2 if and only if, for any quadruple $(x_1, x_2, x_3, x_4) \in [0, 1]^4$, the following holds:

$$T_2\big(T_1(x_1, x_2), T_1(x_3, x_4)\big) \leq T_1\big(T_2(x_1, x_3), T_2(x_2, x_4)\big)$$

It is known [12] that dominance is a reflexive and antisymmetric relation on the set of t-norms, while it is still not clarified whether it is transitive.

One can easily verify that $T_{\mathbf{M}}$ dominates any t-norm T, as well as any t-norm dominates the weakest t-norm $T_{\mathbf{W}}$. Moreover, a t-norm T_1 can only be dominated by another T_2 if T_1 is weaker than T_2, where the reverse does not hold in general.

2.48. Lemma. *Consider two t-norms T_1 and T_2. The T_2-intersection of any two arbitrary T_1-transitive fuzzy relations is T_1-transitive if and only if T_2 dominates T_1.*

Proof. See [12]. □

2.4.2 Congruence and Hulls

The notion of congruence will be especially important for all further investigations. Originally defined for fuzzy equivalence relations under the term "extensionality" (see 2.4.3), we will define a generalization for arbitrary reflexive fuzzy relations. Most of the results are straightforward adaptations of those presented in [34]. Throughout this subsection, R denotes a reflexive fuzzy relation on a domain X and T denotes a left-continuous t-norm.

2.49. Definition. A fuzzy subset $A \in \mathcal{F}(X)$ is called *R-congruent* if and only if

$$\forall x, y \in X: \; T(\mu_A(x), R(x, y)) \leq \mu_A(y).$$

The meaning of congruence is that, for all elements x of A, also all y are contained in A which are in relation with x, i.e. that A is somehow closed under R.

2.50. Lemma. *For any fuzzy set $A \in \mathcal{F}(X)$, R-congruence is equivalent to*

$$\forall x, y \in X: \; R(x, y) \leq \vec{T}(\mu_A(x), \mu_A(y)). \tag{2.10}$$

If R is, in addition, symmetric, A is R-congruent if and only if the following inequality holds:

$$\forall x, y \in X: \; R(x, y) \leq \overset{\leftrightarrow}{T}(\mu_A(x), \mu_A(y)) \tag{2.11}$$

Proof. The first proposition follows directly from the definition of the residuum (cf. Definition 2.28). On the other hand, swapping x and y, we obtain

$$T(\mu_A(y), R(y, x)) \leq \mu_A(x).$$

Taking again the definition of the residuum and the symmetry of R into account, we get

$$\begin{aligned} R(x, y) &\leq \vec{T}(\mu_A(x), \mu_A(y)), \\ R(x, y) &\leq \vec{T}(\mu_A(y), \mu_A(x)). \end{aligned}$$

Hence, the following must hold:

$$R(x, y) \leq \min\left(\vec{T}(\mu_A(x), \mu_A(y)), \vec{T}(\mu_A(y), \mu_A(x))\right) = \overset{\leftrightarrow}{T}(\mu_A(x), \mu_A(y))$$

The opposite direction is, of course, trivial if we consider (2.10). □

The next assertion, which turns out to be extremely helpful later, follows immediately from (2.10).

2.51. Corollary. *Let Q be another reflexive fuzzy relation. If a fuzzy subset A is R-congruent and $Q \leq R$, then A is also Q-congruent.*

The next result clarifies in which way congruence is preserved for $T_{\mathbf{M}}$-intersections, both finite and infinite.

2.52. Lemma. *For a family of R-congruent fuzzy subsets $(C_i)_{i \in I}$, their suprema and infima with respect to inclusion, which are represented by the membership functions*

$$\sup_{i \in I} \mu_{C_i}(x) \qquad\qquad \inf_{i \in I} \mu_{C_i}(x)$$

are also R-congruent. If the index set I is finite, the same holds even if T is not left-continuous.

Proof. For arbitrary $x, y \in X$, we know that

$$T(\mu_{C_i}(x), R(x, y)) \leq \mu_{C_i}(y) \tag{2.12}$$

holds for all $i \in I$ which implies that

$$\sup_{i \in I} T(\mu_{C_i}(x), R(x, y)) \leq \sup_{i \in I} \mu_{C_i}(y).$$

Then left-continuity of T yields

$$T(\sup_{i \in I} \mu_{C_i}(x), R(x, y)) \leq \sup_{i \in I} \mu_{C_i}(y).$$

Inequality (2.12) also implies

$$\inf_{i \in I} T(\mu_{C_i}(x), R(x, y)) \leq \inf_{i \in I} \mu_{C_i}(y),$$

which, together with Lemma 2.13, finally entails

$$T(\inf_{i \in I} \mu_{C_i}(x), R(x, y)) \leq \inf_{i \in I} \mu_{C_i}(y).$$

In the case of a finite index set I, the monotonicity of t-norms is sufficient for the validity of the assertions regardless whether T is left-continuous. □

2.53. Definition. The *hull* with respect to R of a fuzzy set $A \in \mathcal{F}(X)$, denoted $H_R(A)$, is represented by the membership function

$$\mu_{H_R(A)}(x) = \sup\{T(\mu_A(y), R(y, x)) \mid y \in X\}. \tag{2.13}$$

2.54. Lemma. *Provided that R is T-transitive, $H_R(A)$ is the smallest R-congruent superset of A.*

Proof. First of all, reflexivity trivially implies that $A \subseteq H_R(A)$:

$$\mu_{H_R(A)}(x) = \sup\{T(\mu_A(y), R(y, x)) \mid y \in X\} \geq T(\mu_A(x), R(x, x)) = \mu_A(x)$$

For proving that $H_R(A)$ is R-congruent, consider the left-continuity of T and the T-transitivity of R:

$$\begin{aligned}
T(\mu_{H_R(A)}(x), R(x, y)) &= T\big(R(x, y), \sup\{T(\mu_A(z), R(z, x)) \mid z \in X\}\big) \\
&= \sup\{T\big(\mu_A(z), R(z, x), R(x, y)\big) \mid z \in X\} \\
&= \sup\{T\big(\mu_A(z), T(R(z, x), R(x, y))\big) \mid z \in X\} \\
&\leq \sup\{T(\mu_A(z), R(z, y)) \mid z \in X\} \\
&= \mu_{H_R(A)}(y)
\end{aligned}$$

If B is an arbitrary R-congruent superset of A we obtain, for all $y \in X$,

$$\mu_B(x) \geq T(\mu_B(y), R(y,x)) \geq T(\mu_A(y), R(y,x))$$

Hence, we can even take the supremum on the right-hand side, i.e.

$$\mu_B(x) \geq \sup\{T(\mu_A(y), R(y,x)) \mid y \in X\} = \mu_{H_R(A)}(x),$$

which shows that B must be a superset of $H_R(A)$. □

From the above lemma, some fundamental properties can be deduced which we will need frequently when considering hulls with respect to fuzzy orderings in Chapter 5.

2.55. Corollary. *Provided that R is additionally T-transitive, the following propositions hold:*

1. *The hull operation is idempotent: $H_R(H_R(A)) = H_R(A)$*
2. *The operator H_R is monotonic with respect to the inclusion:*

$$A \subseteq B \implies H_R(A) \subseteq H_R(B)$$

3. *H_R can be represented in a dual way:*

$$\mu_{H_R(A)}(x) = \inf\{\mu_B(x) \mid B \text{ is an } R\text{-congruent superset of } A\}$$

2.56. Remark. The notion of congruence and the term "hull" are only meaningful if the relation R is reflexive. However, there is no serious mathematical obstacle not to apply the operation (2.13) even if $R : X \times Y \to [0,1]$ is not reflexive and $X \neq Y$, with the only difference that this formula should rather be interpreted as the image of a fuzzy set A under a fuzzy relation R.

In particular, consider the special case that R represents the graph of a crisp function $f : X \to Y$:

$$R_f(x,y) = \begin{cases} 1 & \text{if } y = f(x) \\ 0 & \text{otherwise} \end{cases}$$

It is easy to check that computing the image with respect to R_f is equivalent to the extension principle (2.9). Even more important, if R can be considered as a relational formulation of a fuzzy knowledge or rule base, the fundamental inference mechanism of approximate reasoning—the so-called *compositional rule of inference*—is obtained [5, 26, 40, 46, 47].

2.4.3 Fuzzy Equivalence Relations

Before we turn to the actual objects of our studies—fuzzy orderings—let us give a brief overview of fuzzy equivalence relations. Fuzzy equivalence relation play an outstanding role in fuzzy set theory as crisp equivalence relations do in the classical theory. The axioms of fuzzy equivalence relations are, more or less, straightforward fuzzifications of the classical three axioms of equivalence relations—reflexivity, symmetry, and transitivity.

Fuzzy equivalence relations were introduced in 1971 by L. A. Zadeh [74] under the name *similarity relations* (only for $T_{\mathbf{M}}$, the generalization to t-norms has been introduced later [63]). This term already gives a clue that they were intended to be models of gradual equality or, more generally, equivalence, as anticipated in Chapter 1. We will use the term *fuzzy equivalence relation* in the following, since it reveals the mathematical motivation behind the axioms in the best way. Other names, which have been used the past 27 years, sometimes in connection with a specific t-norm, are equality relation [33, 35], fuzzy equality [29], indistinguishability relation [29, 63, 64], likeness relation [11], and proximity relation [15].

2.57. Definition. A fuzzy relation E on a domain X is called *fuzzy equivalence relation* with respect to a t-norm T, for brevity *T-equivalence*, if and only if it is reflexive, symmetric, and T-transitive. In addition, E is called *separated* if and only if

$$\forall x, y \in X : \ E(x, y) = 1 \Longleftrightarrow x = y.$$

A separated fuzzy equivalence relation is called *fuzzy equality*.

2.58. Lemma. *Basic relationships:*

1. *Every crisp equivalence relation is a fuzzy equivalence relation with respect to any t-norm. Among crisp equivalence relations, however, the crisp equality is the only one which is separated.*

2. *For every fuzzy equivalence relation, the kernel is a crisp equivalence relation. For any $T_{\mathbf{M}}$-equivalence, each α-cut (both strict and non-strict) is an equivalence relation.*

3. *If $T_1 \leq T_2$, any T_2-equivalence is also a T_1-equivalence.*

Proof. 1. and 3. trivial; 2. see [74]. □

As mentioned above, fuzzy equivalence relations can be regarded as measures of similarity. In functional analysis and topology, (pseudo-)metrics, as they can be considered as measures of distance, are the common concept of similarity. Now we will briefly discuss the relationships between fuzzy equivalence relations and pseudo-metrics.

2.59. Definition. A mapping $d : X^2 \to [0, \infty]$ is called *pseudo-metric* on X if and only if the following axioms hold for all $x, y, z \in X$:

$$\begin{array}{rll} (i) & d(x,x) = 0 & \text{(homogeneity)} \\ (ii) & d(x,y) = d(y,x) & \text{(symmetry)} \\ (iii) & d(x,z) \leq d(x,y) + d(y,z) & \text{(triangle inequality)} \end{array}$$

Moreover, d is called *metric* if strong homogeneity holds:

$$\forall x, y \in X : \; d(x,y) = 0 \Longleftrightarrow x = y$$

The fundamental result, which comes next, establishes construction principles for defining fuzzy equivalence relations from pseudo-metrics and vice versa.

2.60. Theorem. *Let us consider an Archimedean t-norm T with an additive generator f.*

1. *For any pseudo-metric d, the mapping $E_d : X^2 \to [0,1]$ defined as*

$$E_d(x,y) = f^{-1}\big(\min(d(x,y), f(0))\big) \tag{2.14}$$

   is a T-equivalence. E_d is separated if and only if d is a metric.

2. *Provided that E is a T-equivalence on X, we can define a pseudo-metric $d_E : X^2 \to [0, \infty]$ as*

$$d_E(x,y) = f(E(x,y)) \tag{2.15}$$

   which is a metric if and only if E is a fuzzy equality.

Proof. Both assertions have been proved in [11] and [50], where the duality between T-transitivity and the triangle inequality has originally been discovered by S. V. Ovchinnikov [56]. So, we can restrict ourselves to proving the correspondence between strong homogeneity and separability (note that f is strictly increasing and continuous and that $f(1) = 0$):

1. First of all, if $d(x,y)=0$ implies $E_d(x,y)=1$. On the other hand,
$$E_d(x,y)=f^{-1}\big(\min(d(x,y),f(0))\big)=1$$
implies that
$$\min(d(x,y),f(0))=f(1)=0.$$
Finally, we obtain the equivalence
$$E_d(x,y)=1 \iff f^{-1}(d(x,y))=1 \iff d(x,y)=0,$$
which immediately implies that E_d is a fuzzy equality if and only if d is a metric.

2. Similarly to 1., we can deduce from the equivalence
$$d_E(x,y)=0 \iff f(E(x,y))=0 \iff E(x,y)=1$$
that d_E is a metric if and only if E is separated. □

Due to the three axioms, all results provided in 2.4.2 hold for fuzzy equivalence relations if the underlying t-norm is left-continuous. In accordance to the original definition of congruence for fuzzy equivalence relations, let us make the following convention.

2.61. Definition. If E is a fuzzy equivalence relation on X with respect to a left-continuous t-norm T, an E-congruent fuzzy subset A of X will be called *extensional*. Consequently, $H_E(A)$ will be called *extensional hull*, for which we will often use the symbol $\mathrm{EXT}(A)$.

There are a lot of constructions and representations for fuzzy equivalence relations [40, 64]. We will restrict ourselves to those which are significant in our further investigations. An important result is how T-equivalences on a product space can be defined by means of T-equivalences on the component spaces (see [40] for $\tilde{T}=T_{\mathbf{M}}$).

2.62. Theorem. *Let the fuzzy relations $E_1,\dots,E_n$ be T-equivalences on the domains $X_1,\dots,X_n$, respectively. If a t-norm $\tilde{T}$ dominates T, the mapping*
$$\begin{aligned} E:\ (X_1\times\cdots\times X_n)^2 &\longrightarrow [0,1]\\ ((x_1,\dots,x_n),(y_1,\dots,y_n)) &\longmapsto \mathop{\tilde{T}}_{1\le i\le n} E_i(x_i,y_i)\end{aligned}$$
defines a T-equivalence on $X_1\times\cdots\times X_n$. E is separated if and only if all component relations E_i are separated.

Proof. Of course, reflexivity and symmetry are trivial to prove. T-transitivity follows directly from Lemma 2.48 if one considers E as the $\tilde{T}$-intersection of the following "cylindric extensions":

$$\begin{array}{rcl} \tilde{E}_i : \ (X_1 \times \cdots \times X_n)^2 & \longrightarrow & [0,1] \\ ((x_1,\ldots,x_n),(y_1,\ldots,y_n)) & \longmapsto & E_i(x_i,y_i) \end{array}$$

The last assertion

$$\begin{aligned} E((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = 1 &\iff \big(\forall i = 1,\ldots,n : \ E_i(x_i,y_i) = 1\big) \\ &\iff \big(\forall i = 1,\ldots,n : \ x_i = y_i\big) \end{aligned}$$

follows directly if we take the following obvious property of t-norms into account:

$$\forall x,y \in [0,1] : \ \tilde{T}(x,y) = 1 \iff x = y = 1$$

□

It is easy to show that the so-called symmetric kernel of a reflexive and transitive relation (commonly called *preordering*) is an equivalence relation. The fuzzy analogue also holds (see [26, 64] for the special case $\tilde{T} = T$).

2.63. Theorem. *Let R be a reflexive and T-transitive fuzzy relation on X. If a t-norm $\tilde{T}$ dominates T, the following mapping defines a T-equivalence:*

$$E_R(x,y) = \tilde{T}(R(x,y), R(y,x))$$

Proof. Reflexivity follows directly from the reflexivity of R. Symmetry is obvious, while T-transitivity is a direct consequence of Lemma 2.48. □

We have already discussed the example how to define a set of tall people . This example shows one important aspect, which has already been mentioned in Chapter 1, that gradual similarity is an inherent component of fuzziness. The next theorem gives a description how this component of similarity can be "extracted".

2.64. Theorem. *Consider a t-norm T and a family of fuzzy subsets $(A_i)_{i\in I}$ of X. Then the function*

$$\begin{array}{rcl} E : \ X^2 & \longrightarrow & [0,1] \\ (x,y) & \longmapsto & \inf\limits_{i\in I} \overset{\leftrightarrow}{T}(\mu_{A_i}(x), \mu_{A_i}(y)) \end{array}$$

defines a T-equivalence on X. Moreover, E is the greatest T-equivalence on X such that all A_i are extensional.

Proof. See [34, 64]. Note that the maximality is an immediate consequence of Lemma 2.50, Eq. (2.11). □

2.4.4 Fuzzy Orderings

Needless to say, higher mathematics would be unthinkable without ordering relations. As pointed out in Chapter 1, they play a fundamental role even when it concerns fuzzy systems which were introduced with the objective to model human-like decisions by taking the graduality of human thinking and reasoning into account. Since the same graduality appears in the way humans specify preferences, it could be useful to have a model of gradual ordering.

It is near at hand to define fuzzy orderings by taking appropriate fuzzifications of the three classical axioms reflexivity, antisymmetry, and transitivity.

2.65. Definition. A reflexive, T-antisymmetric, and T-transitive binary fuzzy relation is called *fuzzy ordering* with respect to the t-norm T, for brevity *T-ordering.*

The first definition of this type was introduced by Zadeh in 1971 [74] for the minimum t-norm under the name *fuzzy partial ordering.* Accordingly, it is near at hand how to define fuzzy preorderings.

2.66. Definition. A reflexive and T-transitive fuzzy relation is called *fuzzy preordering* with respect to T, for brevity *T-preordering.*

Now it remains to define a criterion for the linearity of a fuzzy ordering. So far, there are three different important notions [18, 55, 74].

2.67. Definition. Let T be a t-norm and let S be a t-conorm.

1. A fuzzy relation R on X is called *weakly linear* if and only if, for each pair $(x, y) \in X^2$, either $R(x, y) > 0$ or $R(y, x) > 0$ holds, equivalently,

$$\max(R(x, y), R(y, x)) > 0.$$

2. A fuzzy relation R on X is called *S-complete* if and only if, for each pair $(x, y) \in X^2$, the following holds:

$$x \neq y \implies S(R(x, y), R(y, x)) = 1$$

   For convenience, we will call a $S_{\mathbf{M}}$-complete fuzzy relation simply *complete.*

3. A fuzzy relation R on X is called *strongly S-complete* if and only if, for each pair $(x, y) \in X^2$, the following holds:

$$S(R(x, y), R(y, x)) = 1$$

Again for convenience, we will call a strongly $S_{\mathbf{M}}$-complete fuzzy relation *strongly complete* or *strongly linear*.

2.68. Lemma. *Basic properties and relationships to the crisp case:*

1. *Every crisp ordering is a fuzzy ordering with respect to any t-norm. A crisp linear ordering is weakly linear, S-complete, and strongly S-complete with respect to any t-conorm S.*

2. *The kernel relation of any fuzzy ordering is an ordering, regardless of the connectives chosen.*

3. *For any $T_{\mathbf{M}}$-ordering R, each α-cut (both strict and non-strict) is an ordering. If R is additionally strongly linear, each α-cut is a linear ordering.*

4. *If a relation R is a fuzzy ordering with respect to some t-norm T, it is also a fuzzy equivalence relation with respect to any t-norm weaker than T.*

Proof. 1., 2. and 4. trivial; 3. see [74]. □

Beside the straightforward fuzzification provided in Definition 2.65, there are considerably many other definitions [18, 55], most of them omitting at least one of the three axioms, some of them using dual properties, such as negative transitivity instead. Among the "classical" axioms, reflexivity is the one which is required least often.

In the book by J. Fodor and M. Roubens (1994) [18], the section on fuzzy orderings starts with the following remark:

> *"All the orderings in this section are T-transitive valued binary relations. However, reflexivity does not hold in general. Instead, antisymmetry is supposed to be satisfied."*

This apodictic statement could be understood as if reflexivity and antisymmetry were conflicting properties which is, without any doubt, not true in the crisp case. Moreover, no further arguments are given why the omission of reflexivity could be desirable. We will leave this discussion to the next chapter. Just for the sake of completeness, Table 2.1 gives an overview of important generalizations of fuzzy orderings.

	reflexive	T-asymmetric	T-antisymmetric	T-transitive	negatively S-transitive	weakly linear	complete	strongly linear
Zadeh (1971)								
fuzzy ordering	—	—	—	$T_{\mathbf{M}}$	—	—	—	—
fuzzy preordering	•	—	—	$T_{\mathbf{M}}$	—	—	—	—
fuzzy partial ordering	•	—	$T_{\mathbf{M}}$	$T_{\mathbf{M}}$	—	—	—	—
fuzzy weak ordering	—	—	—	$T_{\mathbf{M}}$	—	•	—	—
fuzzy linear ordering	—	$T_{\mathbf{M}}$	—	$T_{\mathbf{M}}$	—	•	—	—
Ovchinnikov (1991)								
partial ordering	—	$T_{\mathbf{M}}$	—	$T_{\mathbf{M}}$	—	—	—	—
weak ordering	—	$T_{\mathbf{M}}$	—	—	$S_{\mathbf{M}}$	—	—	—
linear ordering	—	$T_{\mathbf{M}}$	—	—	$S_{\mathbf{M}}$	•	—	—
complete quasi-ordering	—	—	—	$T_{\mathbf{M}}$	—	—	•	—
Fodor and Roubens (1994)								
partial T-preorder	•	—	—	•	—	—	—	—
total T-preorder	•	—	—	•	—	—	—	•
partial T-order	—	—	•	•	—	—	—	—
strict partial T-order	—	•	—	•	—	—	—	—
total T-order	—	—	•	•	—	—	•	—
strict total T-order	—	•	—	•	—	—	•	—

Table 2.1: Various generalized definitions of fuzzy orderings. A bullet entry means that the property corresponding to that column is required for an arbitrary t-norm or t-conorm. Conversely, an entry specifying a certain connective means that the definition applies only to this operation.

Chapter 3

Overcoming the "Crispness" of Fuzzy Orderings

3.1 A Critical View on the Existing Definitions

Without going into detail any further, we mentioned at the end of the previous chapter that there are several generalizations of fuzzy orderings, most of them omitting reflexivity (cf. Table 2.1). In order to find motivations for omitting fundamental properties, such as reflexivity, we will now state some problems which arise when requiring all three classical axioms as in Definition 2.65.

3.1.1 Implications as Orderings?

It is a well-known and often-used fact in mathematical logic that there is a strong connection between implications and orderings. Consider, for instance, the relation

$$\varphi \lesssim \psi \iff (\varphi \to \psi \text{ is a tautology}),$$

where φ and ψ are formulas. It is easy to see that $\lesssim$ defines an ordering of the set of formulas if we always consider two formulas as equal if their evaluations coincide for all interpretations.

Moreover, in the frameworks of many-valued logics based on residuated lattices [25] (including all algebraic structures mentioned in 2.2.4), the correspondence

$$x \leq y \iff (x \to y \text{ is a tautology}) \tag{3.1}$$

holds for arbitrary truth values x and y. The equivalence

$$x \leq y \iff \vec{T}(x, y) = 1 \tag{3.2}$$

(cf. Lemma 2.32) represents just a special case of (3.1), where the underlying structure is $([0, 1], \leq)$ and the operations are t-norm-based. If we consider $\vec{T}$ as a fuzzy relation on the unit interval, (3.2) states that its kernel relation coincides with the crisp linear ordering $\leq$ of the unit interval.

An interesting question is now whether $\vec{T}$ is a fuzzy ordering of the unit interval in the sense of Definition 2.65. The answer is simple: Lemma 2.32 yields that $\vec{T}$ is reflexive and T-transitive. If we assume T-antisymmetry, however, we obtain

$$x \neq y \implies T(\vec{T}(x, y), \vec{T}(x, y)) = 0$$

which actually means that the biimplication $\overleftrightarrow{T}$ lies below the characteristic function of the crisp equality:

$$\chi_{=}(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

Trivially, this is violated for all three basic left-continuous t-norms.

3.1. Proposition. *There is no t-norm T such that any of its residua R satisfies the inequality*

$$T(R(x, y), R(y, x)) \leq \chi_{=}(x, y). \tag{3.3}$$

Proof. First of all, $T(R(x, y), R(y, x)) \leq \chi_{=}(x, y)$ implies that

$$R(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise,} \end{cases}$$

i.e. that R is crisp, which contradicts to $R(1, y) = y$ (cf. Lemma 2.29). □

A trivial consequence of Proposition 3.1 is that there is no t-norm T such that any of its residua is T-antisymmetric. Hence, we obtain that there is no t-norm such that any of its residua can be a fuzzy ordering.

3.1.2 Inclusion Relations

It should be known that, for each non-empty crisp set X, the inclusion $\subseteq$ is an ordering of the power set $\mathcal{P}(X)$. Moreover, $(\mathcal{P}(X), \cap, \cup)$ is a lattice. The same holds in the fuzzy case, i.e. $\subseteq$ is an ordering of the fuzzy power set $\mathcal{F}(X)$ and $(\mathcal{F}(X), \cap_{T_\mathbf{M}}, \cup_{S_\mathbf{M}})$ is a lattice.

We will now try to define a fuzzy concept of inclusion and check whether it can be a fuzzy ordering. In the crisp case, $A \subseteq B$ can be formulated as

$$\forall x \in X :\ x \in A \Longrightarrow x \in B.$$

Fixing a certain left-continuous t-norm T, we can interpret this formula in the setting of fuzzy predicate logic (cf. 2.2.4), even if A and B are fuzzy subsets of X. Then the degree of inclusion can be computed as

$$\mathrm{INCL}_T(A, B) = \inf_{x \in X} \vec{T}(\mu_A(x), \mu_B(x)).$$

Of course, INCL_T can be regarded as a fuzzy relation on $\mathcal{F}(X)$.

3.2. Remark. The relation INCL_T represents just a single variant of a vast number of possible ways to define a gradual concept of inclusion between fuzzy sets. In fact, it is a special case of Bandler and Kohout style fuzzy inclusions [4]

$$\inf_{x \in X} I(\mu_A(x), \mu_B(x)),$$

where I is an arbitrary fuzzy implication (not necessarily a residuum). It is, of course, far beyond the scope of this thesis to give a comprehensive overview of fuzzy inclusions as it was was done, for instance, in [65]; we restrict ourselves to the above example just to demonstrate the difficulties of existing approaches to fuzzy orderings.

3.3. Lemma. *For an arbitrary left-continuous t-norm T, the fuzzy relation INCL_T is a T-preordering on $\mathcal{F}(X)$.*

Proof. Reflexivity follows directly taking the reflexivity of $\vec{T}$ into account (see Lemma 2.32 and Subsection 3.1.1). T-transitivity follows from Lemma 2.13 and the T-transitivity of $\vec{T}$ (cf. Lemma 2.32):

$$\begin{aligned}
&T\big(\mathrm{INCL}_T(A, B), \mathrm{INCL}_T(B, C)\big)\\
&\qquad = T\big(\inf_{x \in X} \vec{T}(\mu_A(x), \mu_B(x)), \inf_{x \in X} \vec{T}(\mu_B(x), \mu_C(x))\big)\\
&\qquad \leq \inf_{x \in X} T\big(\vec{T}(\mu_A(x), \mu_B(x)), \vec{T}(\mu_B(x), \mu_C(x))\big)\\
&\qquad \leq \inf_{x \in X} \vec{T}(\mu_A(x), \mu_C(x))\\
&\qquad = \mathrm{INCL}_T(A, C).
\end{aligned}$$

□

Concerning T-antisymmetry, the answer is again negative. To see that, take an arbitrary element $\tilde{x} \in X$ and define $A = \{\tilde{x}\}$ and B as

$$\mu_B(x) = \begin{cases} r & \text{if } x = \tilde{x}, \\ 0 & \text{otherwise,} \end{cases}$$

with $r \in (0, 1)$. Then the following degrees of inclusion are obtained:

$$\begin{aligned} \mathrm{INCL}_T(A, B) &= \vec{T}(1, r) = r \\ \mathrm{INCL}_T(B, A) &= 1 \end{aligned}$$

Since $T(1, r) = r > 0$, we have shown that there is no t-norm T such that INCL_T is T-antisymmetric. Therefore, INCL_T cannot be a fuzzy ordering, regardless of the t-norm chosen.

3.1.3 The Fuzzification Property

The purpose of this subsection is to introduce a criterion for checking whether a fuzzy relation is a "fuzzification" of a crisp relation. Subsequently, we will apply this criterion to fuzzy equivalence relations and fuzzy orderings.

3.4. Definition. Let $\diamondsuit$ be a crisp relation on a set X. A fuzzy relation R is called $\diamondsuit$-*consistent* if and only if the implication

$$\forall x, y, z \in X : \; y \diamondsuit z \Longrightarrow R(x, y) \leq R(x, z)$$

holds. If, additionally, $\chi_{\diamondsuit} \leq R$ holds, R is called a *fuzzification* of $\diamondsuit$, short "*R fuzzifies $\diamondsuit$*".

3.5. Lemma. *Let R be a fuzzy relation which is $\diamondsuit$-consistent and let $\Box$ be a subrelation of $\diamondsuit$, i.e.*

$$\forall x, y \in X : \; x \Box y \Longrightarrow x \diamondsuit y.$$

Then R is $\Box$-consistent. Furthermore, if R fuzzifies $\diamondsuit$, it also fuzzifies $\Box$.

Proof. Follows directly from the definition. $\Box$

The following result gives a characterization of consistency by means of congruence.

3.6. Proposition. *A reflexive fuzzy relation R is $\diamondsuit$-consistent if and only if every vertical cut $R(x, .)$ is $\diamondsuit$-congruent with respect to any left-continuous t-norm T.*

Proof. For arbitrary $x, y, z \in X$ and an arbitrary left-continuous t-norm T, we can deduce (using Lemmas 2.32 and 2.50):

$$\begin{aligned} T\big(R(x,y), \chi_\diamondsuit(y,z)\big) \leq R(x,z) &\iff \chi_\diamondsuit(y,z) \leq \vec{T}(R(x,y), R(x,z)) \\ &\iff \big(y \diamondsuit z \Longrightarrow R(x,y) \leq R(x,z)\big). \end{aligned} \qquad \square$$

Now we can turn to the first of the two prominent classes—fuzzy equivalence relations. It is not so surprising that they fuzzify their kernel which is a crisp equivalence relation.

3.7. Proposition. *Every T-equivalence E fuzzifies its kernel equivalence relation defined as*

$$x \sim_E y \iff E(x,y) = 1. \tag{3.4}$$

Proof. Due to Lemma 2.58, $\sim_E$ is an equivalence relation. Since, trivially, $\chi_{\sim_E} \leq E$, the only thing to show is $\sim_E$-consistency. If $y \sim_E z$, or equivalently, $E(y,z) = 1$, T-transitivity yields the following:

$$\begin{aligned} E(x,y) &= T(E(x,y), \overbrace{E(y,z)}^{=1}) \leq E(x,z) \\ E(x,z) &= T(E(x,z), \underbrace{E(z,y)}_{=1}) \leq E(x,y) \end{aligned}$$

and we obtain even more:

$$\forall x, y, z \in X: \; y \sim_E z \Longrightarrow E(x,y) = E(x,z) \qquad \square$$

Before considering fuzzifications of orderings, let us take a closer look at the consistency property in the case of a crisp ordering $\lesssim$:

$$\forall x, y, z \in X: \; y \lesssim z \Longrightarrow R(x,y) \leq R(x,z) \tag{3.5}$$

This means that a fuzzy relation R is $\lesssim$-consistent if and only if each vertical cut $R(x,.)$ is non-decreasing, i.e. the degree to which a value y is "smaller or equal" to x is non-decreasing—a property one would naturally demand of a fuzzification of $\lesssim$. The next result, however, shows that there are no non-trivial fuzzifications of crisp linear orderings.

3.8. Proposition. *Let $\lesssim$ be a crisp linear ordering of X. Then $\lesssim$ itself is the only fuzzy ordering which is $\lesssim$-consistent and, as a consequence, the only fuzzification of $\lesssim$.*

Proof. Assume that a fuzzy ordering R is $\lesssim$-consistent. Taking reflexivity and consistency (3.5) into account, we obtain

$$\forall y \gtrsim x : \; R(x, y) = 1.$$

Then T-antisymmetry and linearity imply

$$\forall y \lnsim x : \; R(x, y) = 0$$

and we have shown that $R = \chi_{\lesssim}$. □

This result drastically shows that fuzzy orderings in the sense of Definition 2.65 are not even able to provide consistent fuzzifications of the linear ordering of real numbers.

3.1.4 Reflexivity versus Antisymmetry

It is easy to see from the proof of Proposition 3.8, that the problems in terms of consistency can be avoided if either reflexivity or antisymmetry is dropped. Although this is never mentioned explicitly, one may suspect that the researchers have had the problems, which arise when trying to define fuzzifications of the linear ordering of real numbers, in mind when they proposed non-reflexive fuzzy orderings (as shown in Table 2.1). They went the easier way: Antisymmetry is considered as *the* fundamental property of orderings. Even if reflexivity is omitted, an ordering-like structure (consider, for instance, strict orderings) can be obtained—hence, the conflict between reflexivity and antisymmetry was solved by the omission of reflexivity. On the other hand, the difficulties concerning implications and inclusions are not resolved by dropping reflexivity.

The author is deeply convinced that simply omitting axioms, however, does not solve the problem sufficiently, since the axioms of orderings proved to be appropriate for a long time; every single one has its own justification—omitting just opens the field for arbitrariness. Moreover, it seems to be an eyesore that many properties carry over to the fuzzy variant in the case of equivalence relations, but not in the case of orderings.

Assuming that an approach is desirable, which includes all three classical axioms, but solves all the above problems, let us try to find the actual reasons for the difficulties. Yet the definitions of reflexivity and T-transitivity are, more or less, straightforward. So, we should take a closer look at T-antisymmetry, which is obviously equivalent to (compare with Eq. (3.3))

$$T(R(x, y), R(y, x)) \leq \chi_{=}(x, y). \tag{3.6}$$

One immediately sees that this, indeed, seems to be an appropriate fuzzification of the classical axiom of antisymmetry

$$(x \leq y \wedge y \leq x) \Longrightarrow x = y, \tag{3.7}$$

where the ordering $\leq$ is replaced by the fuzzy ordering R.

Reconsidering the example of the height of people, T-antisymmetry demands, for instance, the following:

$$T(R(179.9, 180.1), R(180.1, 179.9)) = 0$$

This means that two almost indistinguishable heights have to be ranked, more or less, crisply. Asking a human for such an ordering of heights, he/she would naturally take it into account if two people were of about the same height.

In this sense, the definition of T-antisymmetry is a "half-way fuzzification", where the crisp ordering on the left-hand side of (3.7) is replaced by a fuzzy ordering, while the crisp equality on the right-hand side remains untouched.

3.9. Observation. *Evidently, the above example shows that requiring crisp equality in the definition of T-antisymmetry seems to contradict to the nature of vague environments. This is even less surprising if we think of orderings as mathematical models of expressions, such as "smaller/greater or equal", and, consequently, of fuzzy orderings as models of vague expressions, such as "approximately smaller/greater or equal", where one immediately sees the inherent component of similarity. This entails the requirement on fuzzy orderings to take gradual similarity/indistinguishability into account.*

The most obvious way to overcome all the problems seems to replace the crisp equality in (3.6) by a fuzzy concept of equality—a fuzzy equivalence relation. As a consequence, if a fuzzy ordering should respect similarity, the distinction between two values should not be stricter than that provided by the fuzzy equivalence relation. Actually, this means that, following the equivalent formulation of crisp reflexivity

$$\forall x, y \in X: \ x = y \Longrightarrow x \leq y,$$

the crisp equality should also be replaced by a fuzzy equivalence relation.

3.2 Preserving the Classical Axioms by Adding Similarity

According to the discussions of the previous section, we can finally define the similarity-based generalization which will be the main object of investigation throughout the remaining thesis.

3.10. Definition. A T-transitive fuzzy relation $L : X^2 \to [0,1]$ is called *fuzzy ordering* with respect to a t-norm T and a T-equivalence E, for brevity *T-E-ordering*, if and only if it additionally fulfills the following two axioms:

$$\begin{array}{lll} (i) & \forall x,y \in X: \ L(x,y) \geq E(x,y) & (E\text{-reflexivity}) \\ (ii) & \forall x,y \in X: \ T(L(x,y),L(y,x)) \leq E(x,y) & (T\text{-}E\text{-antisymmetry}) \end{array}$$

Before turning to more sophisticated considerations, let us briefly check in which way the above modification relates to the existing concepts of crisp and fuzzy orderings. The following equivalence holds trivially:

$$L(x,x) = 1 \iff L(x,y) \geq \chi_=(x,y)$$

Hence, reflexivity is equivalent to $\chi_=$-reflexivity, while inequality (3.6) states that T-antisymmetry is equivalent to T-$\chi_=$-antisymmetry. We obtain that every T-ordering in the sense of Definition 2.65 fulfills the axioms of Definition 3.10 with $E = \chi_=$. From this point of view, the new definition of T-E-orderings generalizes the existing concept of T-orderings just by admitting an additional degree of freedom—the fuzzy equivalence relation E.

3.2.1 The Interpretation of Induced Similarities

Trivially, a T-E-ordering is a T-preordering. Quite surprisingly, also the reverse is true in some sense.

3.11. Theorem. *Suppose L to be a T-preordering and $\tilde{T}$ to be a t-norm which dominates T. Then L is a fuzzy ordering with respect to T and*

$$E_L(x,y) = \tilde{T}(L(x,y),L(y,x)).$$

Proof. Theorem 2.63 shows that E_L is a T-equivalence. L is obviously E_L-reflexive and T-transitive. The t-norm $\tilde{T}$ can only dominate T if it is stronger (cf. page 39) Hence, we obtain T-E_L-antisymmetry:

$$T(L(x,y),L(y,x)) \leq \tilde{T}(L(x,y),L(y,x)) = E_L(x,y). \qquad \square$$

In particular, the assertion of Theorem 3.11 holds for $\tilde{T} = T$ and $\tilde{T} = T_{\mathbf{M}}$ and we obtain a result which uniquely determines upper and lower bounds for the underlying fuzzy equivalence relation.

3.12. Theorem. *A T-preordering L is a fuzzy ordering with respect to T and a T-equivalence E if and only if, for all $x, y \in X$,*

$$T(L(x,y), L(y,x)) \leq E(x,y) \leq \min(L(x,y), L(y,x)). \tag{3.8}$$

Proof. Obviously, the lower bound

$$T(L(x,y), L(y,x)) \leq E(x,y)$$

directly corresponds to T-E-antisymmetry while the upper bound

$$E(x,y) \leq \min(L(x,y), L(y,x))$$

is equivalent to E-reflexivity. □

3.13. Corollary. *Provided that a T-preordering L is either strongly linear or $T = T_{\mathbf{M}}$ holds,there exists a unique fuzzy equivalence relation E such that L is a T-E-ordering.*

Proof. If L is strongly linear or $T = T_{\mathbf{M}}$, the lower and upper bounds in (3.8) coincide which entails that

$$E(x,y) = \min(L(x,y), L(y,x))$$

is the only T-equivalence such that L is a T-E-ordering. □

Adopting this point of view naively, the new approach seems to result in the hidden removal of the antisymmetry axiom. Yet this is only true if one does not care about the choice of the underlying fuzzy equivalence relation E. If, however, a certain notion of indistinguishability in a certain vague environment is assumed in advance, T-E-antisymmetry has a concrete meaning—that the degree of non-antisymmetry is limited above by the degree of indistinguishability.

The existence of an E such that a T-preordering L is a T-E-ordering only implies that L can be considered as a reasonable concept of ordering if one can consider E as a reasonable concept of indistinguishability in the given environment—otherwise the relation E is of no practical use and its introduction is purely artificial. In this sense, the above two theorems provide criteria for checking whether a given fuzzy preordering has a reasonable interpretation as fuzzy ordering.

In any case, one should not neglect that the same problems can appear even in the crisp case. Considering the example of formulas given in 3.1.1, it is easy to recognize that it is definitely not always a trivial task to specify a proper concept of equality. In many cases, the term "equal" is nonchalantly used when meaning "equivalent". While it is easy to prove that any crisp preordering is antisymmetric up to its symmetric kernel, which is an equivalence relation, a preordering is only acceptable as ordering if the symmetric kernel is an acceptable concept of equality. In which way this—sometimes implicit—factorization can be transferred to the fuzzy case will be studied later.

3.14. Example. In 3.1.1 and 3.1.2, two binary fuzzy relations were discussed, which could be regarded as fuzzy orderings intuitively, but turned out to be only fuzzy preorderings. Now, in the more general framework, Theorem 3.11 guarantees that there are fuzzy equivalence relations such that both can be interpreted as fuzzy orderings. According to the above discussions, it remains to check whether the induced fuzzy equivalence relations are reasonable concepts of indistinguishability.

First of all, for an arbitrary left-continuous t-norm T, we obtain that $\vec{T}$ is indeed a fuzzy ordering with respect to T and

$$T(\vec{T}(x,y),\vec{T}(y,x)) = \overleftrightarrow{T}(x,y).$$

Since, $\overleftrightarrow{T}$ is almost the only imaginable concept of equivalence in logical terms, we see that $\vec{T}$ can be interpreted seriously as a fuzzy ordering. Furthermore, as obvious from Lemma 2.37, Point 2., the biimplication $\overleftrightarrow{T}$ is separated. Therefore, the problems stated in 3.1.1 are perfectly solved in the new framework.

Now let us consider the symmetric kernel of the inclusion relation INCL_T, where, according to Theorem 3.11, we use $\tilde{T} = T_{\mathbf{M}}$:

$$\begin{aligned}
&T_{\mathbf{M}}\big(\mathrm{INCL}_T(A,B),\mathrm{INCL}_T(B,A)\big)\\
&\qquad\qquad = \min\big(\inf_{x\in X}\vec{T}(\mu_A(x),\mu_B(x)),\inf_{x\in X}\vec{T}(\mu_B(x),\mu_A(x))\big)\\
&\qquad\qquad = \inf_{x\in X}\min\big(\vec{T}(\mu_A(x),\mu_B(x)),\vec{T}(\mu_B(x),\mu_A(x))\big)\\
&\qquad\qquad = \inf_{x\in X}\overleftrightarrow{T}(\mu_A(x),\mu_B(x))\big),
\end{aligned}$$

We see that INCL_T induces the fuzzy equivalence relation

$$\mathrm{SIM}_T(A,B) = \inf_{x\in X}\overleftrightarrow{T}(\mu_A(x),\mu_B(x))$$

which is a well-known fuzzy relation for measuring the similarity of fuzzy sets, at least for $T = T_{\mathbf{L}}$ [40, 58, 67]. As a matter of fact, $\mathrm{SIM}_T(A, B)$ turns out to be a fuzzy equality, too:

$$\begin{aligned}
\mathrm{SIM}_T(A,B) = 1 &\iff \inf_{x\in X} \overset{\leftrightarrow}{T}(\mu_B(x), \mu_A(x)) = 1 \\
&\iff \big(\forall x \in X : \ \overset{\leftrightarrow}{T}(\mu_A(x), \mu_B(x)) = 1\big) \\
&\iff \big(\forall x \in X : \ \mu_A(x) = \mu_B(x)\big) \\
&\iff A = B
\end{aligned}$$

So, we have also resolved all the difficulties concerning inclusions.

Chapter 4

Constructions and Representations

The purpose of this chapter is twofold. Firstly, we try to investigate in which way properties of fuzzy orderings are preserved by elementary operations, such as basic set connectives, compositions, or Cartesian products. This study also includes characterizations of inverse and dual relations as well as the attempt to transfer factorization with respect to the underlying equivalence to the fuzzy case. Secondly, a positive answer is given to the question whether the new, generalized framework of fuzzy orderings is able to provide non-trivial fuzzifications of crisp (linear) orderings.

4.1 Applying Connectives to Fuzzy Orderings

4.1.1 Intersections and Unions

It is easy to see that the conjunction of two crisp orderings is again an ordering. This basic fact carries over to the fuzzy case without any serious restrictions.

4.1. Theorem. *Suppose that L_1 is a T-E_1-ordering on X and L_2 is a T-E_2-ordering on X. If $\tilde{T}$ is a t-norm, which dominates T,*

$$L(x,y) = \tilde{T}(L_1(x,y), L_2(x,y))$$

is a T-E-ordering with

$$E(x,y) = \tilde{T}(E_1(x,y), E_2(x,y)).$$

Proof. One easily verifies that E is indeed a T-equivalence [12], where reflexivity and symmetry are trivial, while T-transitivity follows from Lemma 2.48. Trivially, L is E-reflexive simply because of the monotonicity of t-norms. We obtain T-E-antisymmetry as another consequence of dominance:

$$\begin{aligned}
&T\big(\tilde{T}(L_1(x,y),L_2(x,y)),\tilde{T}(L_1(y,x),L_2(y,x))\big)\\
&\qquad\qquad\qquad \le \tilde{T}\big(T(L_1(x,y),L_1(y,x)),T(L_2(x,y),L_2(y,x))\big)\\
&\qquad\qquad\qquad \le \tilde{T}(E_1(x,y),E_2(x,y))\\
&\qquad\qquad\qquad = E(x,y)
\end{aligned}$$

Finally, as above, T-transitivity of L also follows from Lemma 2.48. □

As an immediate consequence of Theorem 4.1, we can deduce that, if $\tilde{T} = T_{\mathbf{M}}$ and $E_1 = E_2$, even the underlying fuzzy equivalence relation is preserved.

4.2. Corollary. *For any two T-E-orderings L_1 and L_2, the intersection with respect to the minimum t-norm*

$$L(x,y) = \min(L_1(x,y),L_2(x,y))$$

is also a T-E-ordering.

The assertions of Theorem 4.1 and Corollary 4.2 hold analogously for any finite intersection of fuzzy orderings, basically because dominance inductively carries over to the n-ary case:

$$T\big(\tilde{T}(x_1,\ldots,x_n),T(y_1,\ldots,y_n)\big) = \tilde{T}\big(T(x_1,y_1),\ldots,T(x_n,y_n)\big)$$

Moreover, Lemma 2.13 provides the basis for a kind of "transfinite" dominance:

$$T(\inf_{i\in I} x_i, \inf_{j\in I} y_j) \le \inf_{i\in I}\inf_{j\in I} T(x_i,y_j) \le \inf_{i\in I} T(x_i,y_i) \tag{4.1}$$

4.3. Corollary. *Let $(L_i)_{i\in I}$ and $(E_i)_{i\in I}$ be two families of fuzzy relations on X such that each L_i is a T-E_i-ordering. Then*

$$L(x,y) = \inf_{i\in I} L_i(x,y)$$

is a T-E-ordering, where

$$E(x,y) = \inf_{i\in I} E_i(x,y).$$

Proof. E-reflexivity is trivial as well as reflexivity and symmetry of E. T-E-antisymmetry and T-transitivity of E and L can be deduced easily with the help of inequality (4.1). □

Computing the union of two fuzzy orderings, however, does not necessarily yield a fuzzy ordering. Trivially, reflexivity is preserved by the union. Yet this is the only of the three properties which generally holds for unions. Consider the following simple counterexample which demonstrates that antisymmetry and transitivity can both be violated even by unions of crisp orderings.

4.4. Example. For $X = \{a, b, c, d\}$, let the two partial orderings $\leq_1$ and $\leq_2$ be defined as follows:

$\leq_1$	a	b	c	d
a	1	0	0	0
b	1	1	1	1
c	0	0	1	1
d	0	0	0	1

$\leq_2$	a	b	c	d
a	1	0	0	0
b	0	1	0	0
c	1	0	1	0
d	1	1	1	1

As easy to see from the Hasse diagrams in Figure 4.1, both relations are partial orderings. The union relation, let us denote it with $\lhd$, is represented by the following table:

$\lhd$	a	b	c	d
a	1	0	0	0
b	1	1	1	1
c	1	0	1	1
d	1	1	1	1

Transitivity is violated, since $c \lhd d$ and $d \lhd b$ but $c \ntriangleleft b$. Obviously, antisymmetry does not hold either because, for example, $c \lhd d$ and $d \lhd c$.

4.1.2 Compositions

Now let us consider what happens if we compute the composition of two fuzzy orderings. The next lemma provides the basis for a quick, but negative answer.

4.5. Lemma. *For any two reflexive fuzzy relations R_1 and R_2 on some domain X, the following inequality holds:*

$$\forall x, y \in X : \; \mu_{R_1 \circ R_2}(x, y) \geq \max(R_1(x, y), R_2(x, y)).$$

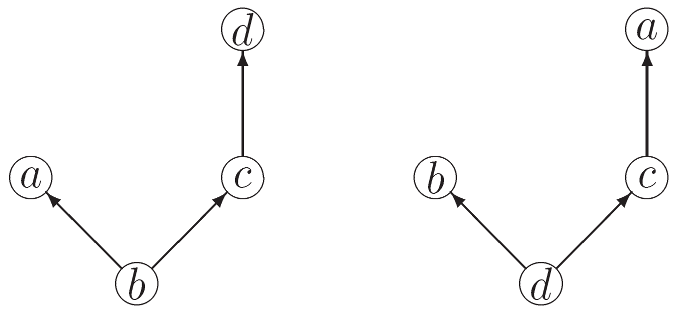


Figure 4.1: Two simple crisp partial orderings the union of which is neither transitive nor antisymmetric. According to Example 4.4, the left graph shows $\leq_1$ while the right graph depicts $\leq_2$. Circles around elements should express that the elements are in relation to themselves.

Proof. Taking reflexivity of both relations into account, we obtain the inequalities

$$\mu_{R_1 \circ R_2}(x,y) = \sup_{z \in X} T(R_1(x,z), R_2(z,y)) \geq T(R_1(x,y), R_2(y,y)) = R_1(x,y),$$
$$\mu_{R_1 \circ R_2}(x,y) = \sup_{z \in X} T(R_1(x,z), R_2(z,y)) \geq T(R_1(x,x), R_2(x,y)) = R_2(x,y),$$

which, together, prove the assertion. □

A trivial consequence of Lemma 4.5 is that reflexivity is again preserved while antisymmetry cannot be satisfied if it is already violated for the $S_{\mathbf{M}}$-union. Hence, the situation concerning antisymmetry in Example 4.4 cannot be better. If we compute the compositions of these two relations[1] $\lhd_1 = \leq_1 \circ \leq_2$ and $\lhd_2 = \leq_2 \circ \leq_1$, the following result is obtained which shows that T-transitivity is not preserved by compositions either:

$\lhd_1$	a	b	c	d
a	1	0	0	0
b	1	1	1	1
c	1	1	1	1
d	1	1	1	1

$\lhd_2$	a	b	c	d
a	1	0	0	0
b	1	1	1	1
c	1	0	1	1
d	1	1	1	1

[1] For finite domains, the composition can be computed efficiently as generalized matrix product, where product and sum have to be replaced by the t-norm and the supremum, respectively.

4.1.3 Cartesian Products

In the crisp case, there are basically two approaches to define orderings of product spaces by means of orderings of the component spaces—Cartesian products, i.e. the conjunction of component relations, and lexicographic composition. While there is not yet a clue how to define fuzzy lexicographical orderings meaningfully for non-trivial cases, the construction of fuzzy orderings by Cartesian products is a straightforward task.

4.6. Theorem. *Let us consider a finite family of crisp sets $(X_1, \dots, X_n)$, an arbitrary t-norm T, and two families of fuzzy relations $(L_1, \dots, L_n)$ and $(E_1, \dots, E_n)$ such that, for all $i \in \{1, \dots, n\}$, E_i is a T-equivalence on X_i and L_i is a T-E_i-ordering on X_i. If a t-norm $\tilde{T}$ dominates T, the mapping*

$$\begin{array}{rcl} \tilde{L}: \quad (X_1 \times \cdots \times X_n)^2 & \longrightarrow & [0,1] \\ \big((x_1, \dots, x_n), (y_1, \dots, y_n)\big) & \longmapsto & \underset{1 \le i \le n}{\tilde{T}} L_i(x_i, y_i) \end{array}$$

is a fuzzy ordering with respect to T and the fuzzy equivalence relation

$$\tilde{E}\big((x_1, \dots, x_n), (y_1, \dots, y_n)\big) = \underset{1 \le i \le n}{\tilde{T}} E_i(x_i, y_i).$$

Proof. Theorem 2.62 already shows that $\tilde{E}$ is a T-equivalence. It is trivial to see that the cylindric extension of every L_i, i.e.

$$\bar{L}_i\big((x_1, \dots, x_n), (y_1, \dots, y_n)\big) = L_i(x_i, y_i)$$

is a fuzzy ordering on $(X_1 \times \cdots \times X_n)$ with respect to T and the cylindric extension of E_i:

$$\bar{E}_i\big((x_1, \dots, x_n), (y_1, \dots, y_n)\big) = E_i(x_i, y_i)$$

Then the result follows directly from Theorem 4.1. □

4.7. Corollary. *Analogously, the assertion of Theorem 4.6 still holds for $\tilde{T} = T_{\mathbf{M}}$ even if the Cartesian product is not finite, i.e.*

$$\tilde{L}\big((x_i)_{i \in I}, (y_i)_{i \in I}\big) = \inf_{i \in I} L_i(x_i, y_i)$$

is a T-$\tilde{E}$-ordering, where

$$\tilde{E}\big((x_i)_{i \in I}, (y_i)_{i \in I}\big) = \inf_{i \in I} E_i(x_i, y_i),$$

if, for all $i \in I$, L_i is a T-E_i-ordering on X_i.

Proof. Consider the cylindric extensions as in the proof of Theorem 4.6 and apply Corollary 4.3. □

4.2 Inverses and Duals

The reader may have observed that the previous considerations excluded one elementary set operation—the complement. As obvious from Definition 2.44, there is a strong connection between the complement, the inverse, and the dual of a fuzzy relation. Before turning to less trivial treatments, let us quickly dispatch the inverse.

4.8. Lemma. *A fuzzy relation L is a T-E-ordering if and only if its inverse L^{-1} is a T-E-ordering. Any of the following properties holds for L if and only if it holds for its inverse: Weak linearity, S-completeness, strong S-completeness, T-S-Ferrers property. Moreover, both lower and upper bounds in (3.8) coincide for L and L^{-1}.*

Proof. Trivial. □

As an important consequence, when considering any of these properties, there is no need to treat complements and duals differently, since a dual relation is nothing else than the inverse of the complement. Hence, all of our basic properties of fuzzy relations hold for the dual if and only if they hold for the complement.

Throughout the remaining section, let (T, S, N) be a de Morgan triple, where the negation N is supposed to be an involution.

Already in the crisp case, not even reflexivity is preserved for the complement or dual. We will, however, not restrict ourselves to listing properties which do not hold; the main objective of the present investigation is to formulate so-called dual properties which hold for the dual if and only if they hold for the original relation. Let us start with the simplest property—reflexivity.

4.9. Proposition. *Suppose that E is a T-equivalence. A fuzzy relation R is E-reflexive if and only if its dual is E-*irreflexive, *i.e.*

$$\forall x, y \in X : \ R^d(x, y) \leq E^d(x, y).$$

Proof. Follows trivially from the monotonicity and bijectivity of N. □

If one considers the value $L(x, y)$, where L is a fuzzy ordering, as the truth value of the fuzzy proposition "x is smaller or equal than y", an intuitive interpretation of E-irreflexivity could be that "x is strictly less than y or x and y are incomparable" always implies that x and y must be distinguishable. As an immediate consequence of the above correspondence, we obtain that a fuzzy relation R is reflexive if and only if R^d is irreflexive.

Considering the dual of a T-E-antisymmetric relation yields a generalized notion of completeness.

4.10. Proposition. *A fuzzy relation R is T-E-antisymmetric if and only if its dual R^d is S-E-*complete*, i.e.*

$$\forall x, y \in X : \; E^d(x,y) \leq S(R^d(x,y), R^d(y,x)).$$

Proof. Taking into account that N is an involution and that (T, S, N) fulfills the first de Morgan law, it follows that T-E-antisymmetry is equivalent to

$$\begin{aligned} E^d(x,y) = N(E(x,y)) &\leq S\big(N(R(x,y)), N(R(y,x))\big) \\ &= S(R^d(x,y), R^d(y,x)). \end{aligned}$$ □

The notion of S-E-completeness can be considered as a straightforward relaxation of S-completeness which also takes indistinguishability into account. As a special case, we obtain from Proposition 4.10 that T-antisymmetry is dual to S-completeness.

Applying the same arguments, one can prove several other important dualities between properties of fuzzy relations. We will summarize them in the next proposition.

4.11. Proposition. *The following holds for any fuzzy relation R:*

1. *R is T-transitive if and only if its dual R^d is negatively S-transitive.*
2. *R is strongly S-complete if and only if the dual relation R^d is T-asymmetric. As a consequence, R is strongly linear if and only if R^d is $T_{\mathbf{M}}$-asymmetric.*

Proof. Both assertions follow straightforwardly from monotonicities and the de Morgan laws [18]. □

Since symmetry is of no further interest for the study of fuzzy orderings[2], the last property to be investigated is the T-S-Ferrers property. As a matter of fact, it appears to be the only one of the properties we consider here except symmetry which is self-dual in general (this correspondence has already been observed by Ovchinnikov [55] for the case $(T, S, N) = (T_{\mathbf{M}}, S_{\mathbf{M}}, N_{\mathbf{S}})$).

4.12. Proposition. *A fuzzy relation R is T-S-Ferrers if and only if R^d is T-S-Ferrers.*

[2]One easily verifies that symmetry is self-dual.

R fulfills		R^d fulfills
reflexivity	$\Longleftrightarrow$	irreflexivity
E-reflexivity	$\Longleftrightarrow$	E-irreflexivity
symmetry	$\Longleftrightarrow$	symmetry
T-antisymmetry	$\Longleftrightarrow$	S-completeness
T-E-antisymmetry	$\Longleftrightarrow$	S-E-completeness
T-transitivity	$\Longleftrightarrow$	negative S-transitivity
strong S-completeness	$\Longleftrightarrow$	T-asymmetry
strong linearity	$\Longleftrightarrow$	$T_{\mathbf{M}}$-asymmetry
T-S-Ferrers property	$\Longleftrightarrow$	T-S-Ferrers property

Table 4.1: An overview of dualities between properties of fuzzy relations.

Proof. If we take arbitrary $a, b, c, d \in X$, the T-S-Ferrers property

$$T(R(a, b), R(c, d)) \leq S(R(a, d), R(c, b))$$

is equivalent to

$$N\big(T(R(a, b), R(c, d))\big) \geq N\big(S(R(a, d), R(c, b))\big).$$

With the de Morgan law and the definition of the dual relation, this is equivalent to

$$S(R^d(b, a), R^d(d, c)) \geq T(R^d(d, a), R^d(b, c)).$$

Substituting x for d, y for a, u for b, and v for c, we obtain

$$T(R^d(x, y), R^d(u, v)) \leq S(R^d(x, v), R^d(u, y))$$

which completes the proof. □

Table 4.1 shows an overview of dualities between the properties we have considered. Since the negation N is supposed to be an involution, $R = (R^d)^d$ always holds and we can swap R and R^d in all above results without any restriction. Hence, duality between properties is always symmetric. Moreover, with Lemma 4.8, we can also replace R^d by the complement in all the above considerations.

Collecting all relevant dualities, we obtain the following characterization of duals of fuzzy orderings.

4.13. Corollary. *A fuzzy relation L is a T-E-ordering if and only if its dual L^d is E-irreflexive, S-E-complete, and negatively S-transitive.*

Since they are of special importance in our further studies, let us consider strongly linear fuzzy orderings in more detail. Immediately, we see that the dual of a strongly linear fuzzy relation is $T_{\mathbf{M}}$-asymmetric. However, we can show even more than that:

4.14. Theorem. *If a T-preordering L is strongly linear, then the following properties hold for both L and L^d: T-transitivity, negative S-transitivity, and T-S-Ferrers.*

Proof. First of all, we prove that L is T-S-Ferrers. For this purpose, let us consider a quadruple $(x, y, u, v) \in X^4$. Since L is assumed to be strongly linear, either $L(x, u) = 1$ or $L(u, x) = 1$ must hold. Assume $L(x, u) = 1$ and we can deduce the following:

$$T(L(x, y), L(u, v)) \leq T(L(x, u), L(u, v)) \leq L(x, v) \leq S(L(x, v), L(u, y))$$

Analogous for $L(u, x) = 1$:

$$T(L(x, y), L(u, v)) \leq T(L(u, x), L(x, y)) \leq L(u, y) \leq S(L(x, v), L(u, y))$$

Due to Proposition 4.12, the T-S-Ferrers property carries over to the dual L^d automatically.

It only remains to show that L is negatively S-transitive—all the other assertions are immediate consequences of Proposition 4.11.

Negative S-transitivity, however, is easy to prove if we take reflexivity and the T-S-Ferrers property into account (compare with [18, Prop. 2.37]):

$$L(x, z) = T(L(x, z), L(y, y)) \leq S(L(x, y), L(y, z)) \qquad \square$$

4.15. Example. Note that, if a fuzzy ordering L is not strongly linear, it cannot be guaranteed neither that L is negatively S-transitive nor that L^d is transitive. Consider the example of a crisp pentagon as shown in Figure 4.2. This ordering and its dual, denoted $\leq$ and $<'$, respectively, are represented by the following tables:

$\leq$	a	b	c	d	e
a	1	1	1	1	1
b	0	1	1	0	1
c	0	0	1	0	1
d	0	0	0	1	1
e	0	0	0	0	1

$<'$	a	b	c	d	e
a	0	1	1	1	1
b	0	0	1	1	1
c	0	0	0	1	1
d	0	1	1	0	1
e	0	0	0	0	0

We see that $b \leq c$ but neither $b \leq d$ nor $d \leq c$, therefore $\leq$ is not negatively S-transitive, regardless which de Morgan triple we have chosen. Moreover, $c <' d$ and $d <' b$, but $c \not<' b$, which implies that the dual $<'$ cannot be T-transitive.

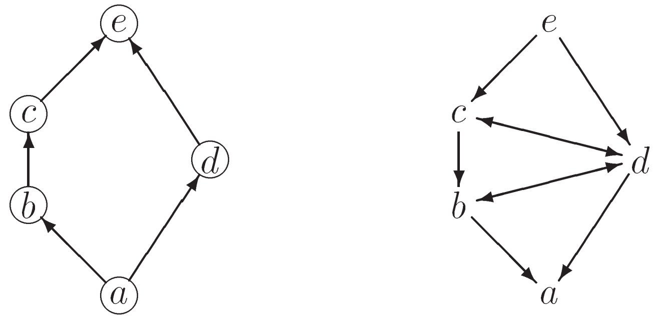


Figure 4.2: Hasse diagrams for the pentagon ordering and its dual. Elements are circled if and only if they are in relation to themselves.

Theorem 4.14 shows that strong linearity is a sufficient condition for a T-preordering to fulfill the T-S-Ferrers property and negative S-transitivity. Finally, we can show that strong S-completeness is even a necessary and sufficient condition for the assertions of Theorem 4.14 to be satisfied, at least if the considered t-norm T is continuous.

In order to show that, we need an important prerequisite the proof of which can be found in [18].

4.16. Lemma. *Suppose that T is continuous. If a fuzzy relation R is T-asymmetric and negatively S-transitive then R is T-S-Ferrers.*

4.17. Theorem. *Let T be a continuous t-norm. Then the following three statements are equivalent for any T-preordering L:*

(i) L is strongly S-complete.

(ii) L is T-S-Ferrers.

(iii) L is negatively S-transitive.

Proof. (i)⇒(ii): If L is strongly S-complete, its dual L^d is T-asymmetric. Since L is T-transitive, L^d must be negatively S-transitive and Lemma 4.16 entails that L^d is T-S-Ferrers. Due to Proposition 4.12, L is T-S-Ferrers, too.

(ii)⇒(iii): As mentioned in the proof of Theorem 4.14, negative S-transitivity follows directly from reflexivity and the T-S-Ferrers property:

$$L(x,z) = T(L(x,z), L(y,y)) \leq S(L(x,y), L(y,z))$$

(iii)⇒(i): Reflexivity and negative S-transitivity immediately imply strong S-completeness

$$1 = L(x,x) \leq S(L(x,y), L(y,x))$$

which completes the proof. □

4.3 Factorization

We have already mentioned in 3.2.1 that factorization offers a way to make a preordering an ordering of a factor space. More specifically, consider a preordering $\preceq$ and an equivalence relation $\sim$ such that the following holds:

$$\begin{array}{ll} (i) & \forall x,y \in X: \ x \sim y \Longrightarrow x \preceq y \\ (ii) & \forall x,y \in X: \ (x \preceq y \wedge y \preceq x) \Longrightarrow x \sim y \end{array}$$

Then $\preceq$ is an ordering of the factor space $X_{/\sim}$, where (i) guarantees that the projection onto $X_{/\sim}$ is well-defined and (ii) directly implies antisymmetry.

Now the question arises in which way this can be transferred to the fuzzy case. If we replace $\sim$ by a fuzzy equivalence relation E, $\preceq$ by some fuzzy relation L, and the logical connectives by their fuzzy equivalents, (i) is nothing else than E-reflexivity while (ii) directly corresponds to T-E-antisymmetry. Basically, there are two ways how to perform this generalized kind of factorization. The first variant is based on the factorization with respect to the kernel relation of the fuzzy equivalence relation E.

4.18. Theorem. *If L is a T-E-ordering on a domain X, it must be a T-E-ordering on $X_{/\sim_E}$ as well, where $\sim_E$ is defined as in (3.4). Moreover, E is a fuzzy equality on $X_{/\sim_E}$.*

Proof. First, we have to show that the projections of E and L are well-defined, i.e.

$$\begin{array}{l} \forall x,y,x',y' \in X: \ \big(x \sim_E x' \wedge y \sim_E y'\big) \Longrightarrow E(x,y) = E(x',y'), \\ \forall x,y,x',y' \in X: \ \big(x \sim_E x' \wedge y \sim_E y'\big) \Longrightarrow L(x,y) = L(x',y'). \end{array}$$

The first assertion has already been show in the proof of Proposition 3.7. For showing that L can be projected in a well-defined way, take into account that the following implications are immediate consequences of E-reflexivity:

$$\begin{array}{rcl} x \sim_E x' & \Longrightarrow & L(x,x') = L(x',x) = 1 \\ y \sim_E y' & \Longrightarrow & L(y,y') = L(y',y) = 1 \end{array}$$

Thus, we can deduce

$$\begin{aligned}
L(x,y) &= T(L(x',x),L(x,y)) \\
&\le L(x',y) = T(L(x',y),L(y,y') \le L(x',y'), \\
L(x',y') &= T(L(x,x'),L(x',y')) \\
&\le L(x,y') = T(L(x,y'),L(y',y) \le L(x,y).
\end{aligned}$$

Reflexivity, E-reflexivity, symmetry, T-E-antisymmetry and T-transitivity automatically carry over to the factor space. The only thing, which remains to show, is that E is separated on $X_{/\sim_E}$:

$$E(\langle x\rangle,\langle y\rangle)=1 \iff E(x,y)=1 \iff x\sim_E y \iff \langle x\rangle=\langle y\rangle \qquad \square$$

The second variant relies on factorization with respect to E itself (compare with [26, Prop. 2.33]). Since we have been lacking the definition of a fuzzification of equivalence classes, let us supply it now.

4.19. Definition. Suppose that E is a T-equivalence on X. The *fuzzy equivalence class* $\langle x_0\rangle$ of an element $x_0\in X$ with respect to E is a fuzzy subset of X which is represented by the membership function

$$\mu_{\langle x_0\rangle}(x)=E(x_0,x).$$

The *fuzzy factor set*—the set system of all fuzzy equivalence classes—is denoted as follows:

$$X_{/E}=\{\langle x\rangle \mid x\in X\}\subseteq \mathcal{F}(X)$$

4.20. Theorem. *Consider a T-equivalence E and a T-E-ordering L on a domain X. Then the projection of E onto $X_{/E}$*

$$\tilde{E}(\langle x\rangle,\langle y\rangle)=E(x,y)$$

is a fuzzy equality with respect to T on $X_{/E}$ and the projection of L

$$\tilde{L}(\langle x\rangle,\langle y\rangle)=L(x,y)$$

is a T-$\tilde{E}$-ordering on $X_{/E}$.

Proof. Again, the first thing to show is well-definedness of the projections:

$$\forall x,y,x',y'\in X:\ \big(\langle x\rangle=\langle x'\rangle\wedge\langle y\rangle=\langle y'\rangle\big)\Longrightarrow \tilde{E}(\langle x\rangle,\langle y\rangle)=\tilde{E}(\langle x'\rangle,\langle y'\rangle)$$
$$\forall x,y,x',y'\in X:\ \big(\langle x\rangle=\langle x'\rangle\wedge\langle y\rangle=\langle y'\rangle\big)\Longrightarrow \tilde{L}(\langle x\rangle,\langle y\rangle)=\tilde{L}(\langle x'\rangle,\langle y'\rangle)$$

As obvious from the definition, $\langle x \rangle = \langle x' \rangle$ is equivalent to

$$\forall y \in X: \; E(x,y) = E(x',y).$$

Assigning x to y, we obtain $E(x,x') = 1$. Conversely, $E(x,x') = 1$ implies

$$\begin{aligned} E(x,y) &= T(E(x',x), E(x,y)) \leq E(x',y), \\ E(x',y) &= T(E(x,x'), E(x,y)) \leq E(x,y). \end{aligned}$$

So, we have proved that the following equivalence holds:

$$\langle x \rangle = \langle x' \rangle \iff E(x,x') = 1 \iff \tilde{E}(\langle x \rangle, \langle x' \rangle) = 1. \tag{4.2}$$

Now the same arguments as in the proof of Theorem 4.18 can be applied to prove well-definedness of all projections. Furthermore, (4.2) already proves that $\tilde{E}$ is separated on $X_{/E}$. The other properties, of course, transfer automatically to the projections. □

4.4 The Fuzzification Property Revisited

Example 3.14 has demonstrated that already two of the three problems stated in 3.1 are solved in the generalized framework of fuzzy orderings. It remains to investigate in which way the new class of fuzzy orderings is able to provide fuzzifications of crisp, in particular, linear orderings.

4.4.1 Extracting Crisp From Fuzzy Orderings

In this subsection, we discuss in which way fuzzy orderings fuzzify crisp orderings. To achieve this goal, we have to study kernel relations of fuzzy orderings beforehand.

4.21. Lemma. *For every T-E-ordering L, its kernel relation*

$$x \trianglelefteq_L y \iff L(x,y) = 1$$

defines a preordering. Moreover, $\trianglelefteq_L$ is an ordering if and only if E is separated. If E is separated, $\trianglelefteq_L$ is a linear ordering if and only if L is strongly linear.

Proof. Reflexivity follows directly from the E-reflexivity of L:

$$1 \geq L(x,x) \geq E(x,x) = 1 \implies x \trianglelefteq_L x$$

In order to prove transitivity, consider the equivalences

$$\begin{aligned} x \trianglelefteq_L y &\iff L(x,y) = 1, \\ y \trianglelefteq_L z &\iff L(y,z) = 1, \end{aligned}$$

and T-transitivity entails

$$1 = T(L(x,y), L(y,z)) \leq L(x,z) \leq 1 \implies x \trianglelefteq_L z.$$

For proving that $\trianglelefteq_L$ is antisymmetric if and only if E is a fuzzy equality, assume first that, for a pair (x,y), both inclusions hold:

$$\begin{aligned} x \trianglelefteq_L y &\iff L(x,y) = 1 \\ y \trianglelefteq_L x &\iff L(y,x) = 1 \end{aligned}$$

Then T, E-antisymmetry implies that

$$1 = T(L(x,y), L(y,x)) \leq E(x,y) \leq 1 \implies E(x,y) = 1$$

If E is separated this implies that x and y must be equal.

Reversely, suppose that E is not separated. Then there are two different values $x, y \in X$ such that $E(x,y) = 1$ and E-reflexivity implies

$$\begin{aligned} 1 = E(x,y) \leq L(x,y) \leq 1 &\implies x \trianglelefteq_L y \\ 1 = E(y,x) \leq L(y,x) \leq 1 &\implies y \trianglelefteq_L x \end{aligned}$$

which contradicts to the antisymmetry of $\trianglelefteq_L$.

Finally, assume that $\trianglelefteq_L$ is an ordering. For arbitrary $x, y \in X$, we obtain

$$(x \trianglelefteq_L y \vee y \trianglelefteq_L x) \iff (L(x,y) = 1 \vee L(y,x) = 1)$$

which completes the proof. □

4.22. Proposition. *A T-E-ordering L fuzzifies its kernel $\trianglelefteq_L$ and any crisp ordering which is a subrelation of $\trianglelefteq_L$.*

Proof. Trivially, the characteristic function of $\trianglelefteq_L$ is below L. Therefore, it is sufficient to show $\trianglelefteq_L$-consistency:

$$\forall x, y, z : \; y \trianglelefteq_L z \Longrightarrow L(x,y) \leq L(x,z)$$

Let $x, y, z \in X$ be arbitrary but fixed such that $y \trianglelefteq_L z$ which is equivalent to $L(y,z) = 1$. Then T-transitivity directly implies

$$L(x,y) = T(L(x,y), L(y,z)) \leq L(x,z).$$

If $\preceq$ is an ordering which is a subrelation of $\trianglelefteq_L$, i.e.

$$\forall x, y \in X : \; x \preceq y \Longrightarrow x \trianglelefteq_L y,$$

it follows from Lemma 3.5 that L fuzzifies $\preceq$, too. □

We already know from Lemma 4.21 that $\trianglelefteq_L$ is not guaranteed to be an ordering for all fuzzy orderings L. From Proposition 4.22, we have obtained that a fuzzy ordering fuzzifies any ordering contained in its kernel. However, we still do not know whether such orderings exist at all or how they can be constructed. So, there are two questions remaining to be clarified:

1. Is there a way to find non-trivial crisp orderings which are contained in the kernel relation $\trianglelefteq_L$ of a fuzzy ordering L?
2. If so, in which way does such a crisp ordering interact with the "fuzzy part" of L—the areas where $L(x, y) \in (0, 1)$?

We will see soon that the there is a quite intuitive, but not necessarily constructive way to answer the first question. In order to clarify the second question, we need a prerequisite—the notion of compatibility.

4.23. Definition. Let $\lesssim$ be a crisp ordering on X and let E be a fuzzy equivalence relation on X. E is called *compatible with* $\lesssim$, if and only if the following implication holds for all $x, y, z \in X$:

$$x \lesssim y \lesssim z \Longrightarrow (E(x, z) \leq E(y, z) \wedge E(x, z) \leq E(x, y)) \tag{4.3}$$

Although this seems to be a purely technical property, there is rather an intuitive interpretation: The two outer elements of a three-element chain cannot be less distinguishable than any two inner elements. Furthermore, there are two other interpretations—one establishing a connection convexity and a characterization of a dual property for pseudo-metrics corresponding to E.

4.24. Proposition. *A fuzzy equivalence relation E is compatible with an ordering $\lesssim$ if every equivalence class $\langle x_0 \rangle$ is convex. If $\lesssim$ is linear the reverse holds as well.*

Proof. Assume that every $\langle x_0 \rangle$ is convex, i.e.

$$\forall x, y, z \in X: \ x \lesssim y \lesssim z \Longrightarrow E(x_0, y) \geq \min(E(x_0, x), E(x_0, z)).$$

With the setting $x_0 = z$, convexity implies that

$$E(x, z) = \min(E(x, z), E(z, z)) \leq E(y, z).$$

Analogously, if we assign x to x_0 we obtain

$$E(x, z) = \min(E(x, x), E(x, z)) \leq E(x, y)$$

and we have proved that compatibility follows from the convexity of all equivalence classes.

Reversely, assume $\lesssim$ to be linear and E to be compatible with $\lesssim$. For an arbitrary $x_0 \in X$ and a sequence $x \lesssim y \lesssim z$, we can distinguish between the following two cases in order to prove convexity of $\langle x_0 \rangle$:

1. $x_0 \lesssim y$: Hence, $x_0 \lesssim y \lesssim z$ and compatibility implies
$$E(x_0, y) \geq E(x_0, z).$$
2. $x_0 \gnsim y$: Here, $x \lesssim y \lesssim x_0$, and we obtain
$$E(x_0, y) \geq E(x_0, x)$$

which completes the proof. □

4.25. Proposition. *Let T be a continuous Archimedean t-norm with an additive generator f and let $\lesssim$ be an ordering of the domain X.*

1. *If a pseudo-metric d on X has the property*
$$\forall x, y, z \in X: \; x \lesssim y \lesssim z \Longrightarrow d(x, z) \geq \max(d(x, y), d(y, z)), \tag{4.4}$$
then its induced fuzzy equivalence relation E_d, defined as in (2.14), is compatible with $\lesssim$.
2. *If a fuzzy equivalence relation E is compatible with $\lesssim$, its induced pseudo-metric d_E, defined as in (2.15), fulfills property (4.4).*

Proof. Follows directly from the fact that the additive generator f and its inverse are non-increasing function. □

Obviously, the interpretation of property (4.4) is that the two outer elements of an ordered three-element chain cannot be closer than any two inner elements.

Now we can turn to *the* fundamental result of the present investigations. It provides a way how to construct maximal crisp orderings which are contained in a given fuzzy ordering.

4.26. Theorem. *Provided that L is a T-E-ordering, there exists a crisp ordering $\lesssim$, such that E is compatible with $\lesssim$ and the following implication holds:*
$$x \lesssim y \implies L(x, y) = 1 \tag{4.5}$$

If L is a strongly linear then $\lesssim$ can be chosen such that it is a linear ordering. Moreover, $\lesssim$ is maximal in the sense that there is no ordering $\lesssim'$ such that $\chi_{\lesssim'} \leq L$ and

$$\{(x, y) \mid x \lesssim y\} \subset \{(x, y) \mid x \lesssim' y\}.$$

Proof. We already know from Lemma 4.21 that the kernel relation $\trianglelefteq_L$ is always a preordering. Now, for all $x, y \in X$, we define a relation $\sim$ as the symmetric kernel of $\trianglelefteq_L$ which is, of course, an equivalence relation:

$$x \sim y \iff (L(x, y) = 1 \wedge L(y, x) = 1) \iff (x \trianglelefteq_L y \wedge y \trianglelefteq_L x).$$

The well-ordering theorem [6, 41] states that every set can be ordered linearly. Thus, it is possible to find a linear ordering of all equivalence classes with respect to $\sim$. For any equivalence class $\langle x \rangle$, let us denote this ordering with $\lesssim_x$.

This enables us to define an ordering $\lesssim$ by means of lexicographic composition of $\trianglelefteq_L$ and all $\lesssim_x$:

$$x \lesssim y \iff \big((x \sim y \wedge x \lesssim_x y) \vee (x \not\sim y \wedge x \trianglelefteq_L y)\big) \tag{4.6}$$

The reflexivity of this relation follows directly from the fact that the involved relations $\sim$, $\trianglelefteq_L$, and all $\lesssim_x$ are reflexive. Moreover, it is easy to see directly from the definition above that $x \trianglelefteq_L y$ is a necessary condition for $x \lesssim y$ which proves (4.5).

In order to prove transitivity, let us consider an arbitrary triple (x, y, z) fulfilling $x \lesssim y$ and $y \lesssim z$. We can distinguish between the following four cases:

1. $x \sim y \wedge y \sim z$: In this case, all three elements x, y, and z belong to the same equivalence class. Therefore, $x \lesssim z$ follows from the transitivity of $\lesssim_x$.

2. $x \sim y \wedge y \not\sim z$: First of all, $x \sim z$ cannot be fulfilled, since this would contradict to the transitivity of $\sim$. We know that $L(y, z) = 1$, $L(x, y) = 1$, and $L(y, x) = 1$ (cf. definition of $\trianglelefteq_L$ and $\sim$) and we obtain

   $$1 \geq L(x, z) \geq T(L(x, y), L(y, z)) = 1,$$

   which immediately yields $x \lesssim z$.

3. $x \not\sim y \wedge y \sim z$: Analogous to (ii).

4. $x \not\sim y \wedge y \not\sim z$: Here we have that $x \trianglelefteq_L y$ and $y \trianglelefteq_L z$, but neither $y \trianglelefteq_L x$ nor $z \trianglelefteq_L y$. It is sufficient to show that $x \trianglelefteq_L z$ but $z \not\trianglelefteq_L x$. The assertion $x \trianglelefteq_L z$ follows directly from the transitivity of $\trianglelefteq_L$. Now assume that $z \trianglelefteq_L x$. Together with $y \trianglelefteq_L z$, the transitivity would yield $y \trianglelefteq_L x$ which is a contradiction.

Now let us consider antisymmetry. Assume that there is a pair (x, y) such that both inequalities $x \lesssim y$ and $y \lesssim x$ hold. With (4.5) we obtain $x \sim y$ and the antisymmetry of the ordering $\lesssim_x$ implies that x and y must be equal.

Provided that L is strongly linear, we know that, for any pair (x, y), either $x \trianglelefteq_L y$ or $y \trianglelefteq_L x$ must hold. In the case $x \not\sim y$ this is already sufficient for linearity. On the other hand, if $x \sim y$, linearity follows directly from the fact that every $\lesssim_x$ was chosen as linear ordering.

In order to prove maximality, suppose that there is a crisp ordering $\lesssim'$ whose characteristic function is between the one of $\lesssim$ and L, i.e.

$$\forall x, y \in X: \ \chi_{\lesssim}(x, y) \le \chi_{\lesssim'}(x, y) \le L(x, y). \tag{4.7}$$

If we assume that $\lesssim$ and $\lesssim'$ are not equal, which means that there are two different elements $\bar{x}$ and $\bar{y}$ such that $\bar{x} \lesssim' \bar{y}$ but $\bar{x} \not\lesssim \bar{y}$, we can distinguish between the following two cases:

1. If $\bar{x} \sim \bar{y}$, the linearity of $\lesssim_x$ implies that, since $\bar{x} \not\lesssim \bar{y}$, the inequality $\bar{y} \lesssim \bar{x}$ must hold. Hence, we obtain from the inequality (4.7) that $\bar{y} \lesssim' \bar{x}$ which contradicts to antisymmetry.

2. If, however, $\bar{x} \not\sim \bar{y}$ then (4.7) implies $L(\bar{x}, \bar{y}) = 1$ which is equivalent to $\bar{x} \trianglelefteq_L \bar{y}$. Then, since $\bar{x} \not\sim \bar{y}$, the inequality $\bar{x} \lesssim \bar{y}$ has to be valid, which is again a contradiction.

It remains to show the compatibility of E with $\lesssim$. From (4.5) we know that $x \lesssim y \lesssim z$ implies $L(x, y) = L(y, z) = L(x, z) = 1$ and we obtain

$$\begin{aligned} E(y, z) &\ge T(L(y, z), L(z, y)) = L(z, y) \\ &\ge T(L(z, x), L(x, y)) = L(z, x) \ge E(x, z), \\ E(x, y) &\ge T(L(x, y), L(y, x)) = L(y, x) \\ &\ge T(L(y, z), L(z, x)) = L(z, x) \ge E(x, z), \end{aligned}$$

which finally completes the proof. □

For a fuzzy ordering L, we have constructed a maximal ordering $\lesssim$ which is contained in the kernel relation $\trianglelefteq_L$. Since the fuzzy ordering L always fuzzifies its kernel relation, it automatically follows from Lemma 3.5 that it also fuzzifies $\lesssim$. The resulting compatibility characterizes the interaction between L, E, and $\lesssim$.

4.4.2 Direct Fuzzifications of Crisp Orderings

In accordance to the difficulties stated in 3.1.3, the next step is to apply the above results to the special case of strongly linear fuzzy orderings.

4.27. Theorem. *Let L be a binary fuzzy relation on X and let E be a T-equivalence on X. Then the following two statements are equivalent:*

(i) L is a strongly linear T-E-ordering on X.

(ii) There exists a linear ordering $\lesssim$ the relation E is compatible with such that L can be represented as follows:

$$L(x,y) = \begin{cases} 1 & \text{if } x \lesssim y \\ E(x,y) & \text{otherwise} \end{cases} \tag{4.8}$$

Proof. (i)⇒(ii): We know from Theorem 4.26 that the relation defined in (4.6) is a crisp linear ordering of X the relation E is compatible with. If $x \lesssim y$, the equation $L(x,y) = 1$ must be fulfilled. Now assume that $x \not\lesssim y$. Since $\lesssim$ is linear, this implies $y \lesssim x$ and we obtain

$$E(x,y) \leq L(x,y) = T(L(x,y), \underbrace{L(y,x)}_{=1}) \leq E(x,y),$$

which shows that $L(x,y) = E(x,y)$.

(ii)⇒(i): It is sufficient to show that Equation (4.8) defines a strongly linear fuzzy ordering.

E-reflexivity follows immediately from the definition. For proving T, E-antisymmetry, without loss of generality, assume $x \lesssim y$. Then the equalities $L(x,y) = 1$ and $L(y,x) = E(x,y)$ hold and we obtain

$$T(\underbrace{L(x,y)}_{=1}, L(y,x)) = L(y,x) = E(x,y).$$

The same argument can be applied analogously in the case $y \lesssim x$.

In order to prove T-transitivity, we have to distinguish between the following cases:

1. $x \lesssim y \lesssim z$, $x \lesssim z \lesssim y$, or $y \lesssim x \lesssim z$: Since $L(x,z)$ is always 1, transitivity can never be violated.
2. $z \lesssim y \lesssim x$: Here, $L(x,y) = E(x,y)$, $L(y,z) = E(y,z)$, and $L(x,z) = E(x,z)$ and T-transitivity follows directly from the T-transitivity of E.
3. $z \lesssim x \lesssim y$: In this case, $L(x,y) = 1$, $L(y,z) = E(y,z)$, and $L(x,z) = E(x,z)$. The compatibility of $\lesssim$ and E implies that $E(y,z) \leq E(x,z)$. Thus, we obtain

$$T(L(x,y), L(y,z)) = T(1, E(y,z)) = E(y,z) \leq E(x,z) = L(x,z).$$

4. $y \lesssim z \lesssim x$: Analogous to 3.

Strong linearity of L follows directly from the linearity of $\lesssim$. □

Theorem 4.27 states that strongly linear fuzzy orderings are uniquely characterized as fuzzifications of crisp linear orderings, where the fuzziness can be attributed to the underlying fuzzy equivalence relation.

4.28. Example. Consider the real numbers $X = \mathbb{R}$. It is easy to check that $d(x,y) = |x - y|$ is a metric which is compatible with the ordinary linear ordering of real numbers $\leq$ in the sense of (4.4). Taking into account that $1 - x$ is a self-inverse additive generator of $T_{\mathbf{L}}$ and that $-\ln x$ is an additive generator of $T_{\mathbf{P}}$ whose inverse is e^{-x}, we obtain two fuzzy equivalence relations which are compatible with $\leq$ (guaranteed with Theorem 2.60 and Proposition 4.25):

$$E_1(x,y) = 1 - (\min(|x-y|, 1)) = \max(1 - |x-y|, 0)$$
$$E_2(x,y) = e^{-\min(|x-y|,\infty)} = e^{-|x-y|}$$

Moreover, the compatibility between an ordering and a metric remains untouched even if a monotonic, bijective transformation is performed. If we introduce two bijective mappings φ and ψ, defined as

$$\varphi(x) = \begin{cases} x & \text{if } x \leq 2, \\ 3x - 4 & \text{if } x \in (2,4), \\ x + 4 & \text{otherwise,} \end{cases}$$
$$\psi(x) = \begin{cases} 3 - \sqrt{4 - (x-1)^2} & \text{if } x \in (1,3), \\ x & \text{otherwise,} \end{cases}$$

we obtain another four fuzzy equivalence relations:

$$
\begin{aligned}
E_3(x,y) &= \max(1 - |\varphi(x) - \varphi(y)|, 0) \\
E_4(x,y) &= e^{-|\varphi(x)-\varphi(y)|} \\
E_5(x,y) &= \max(1 - |\psi(x) - \psi(y)|, 0) \\
E_6(x,y) &= e^{-|\psi(x)-\psi(y)|}
\end{aligned}
$$

Utilizing the representation (4.8), we are able to define six strongly linear fuzzy orderings $L_1, \ldots, L_6$, where L_1 is a $T_{\mathbf{L}}$-E_1-ordering, L_2 is a $T_{\mathbf{P}}$-E_2-ordering, L_3 is a $T_{\mathbf{L}}$-E_3-ordering, L_4 is a $T_{\mathbf{P}}$-E_4-ordering, L_5 is a $T_{\mathbf{L}}$-E_5-ordering, and L_6 is a $T_{\mathbf{P}}$-E_6-ordering. The shapes of these six relations are visualized in Figure 4.3.

Of course, it would be nice to have that the assertion of Theorem 4.27 still holds if we omit any assumptions concerning linearity. The following example, however, shows that this not true in general.

4.29. Example. Consider a four-element set $X = \{a, b, c, d\}$ with the following two relations:

$\leq$	a	b	c	d
a	1	1	1	1
b	0	1	0	1
c	0	0	1	1
d	0	0	0	1

E	a	b	c	d
a	1	0.5	0.5	0
b	0.5	1	0	0.5
c	0.5	0	1	0.5
d	0	0.5	0.5	1

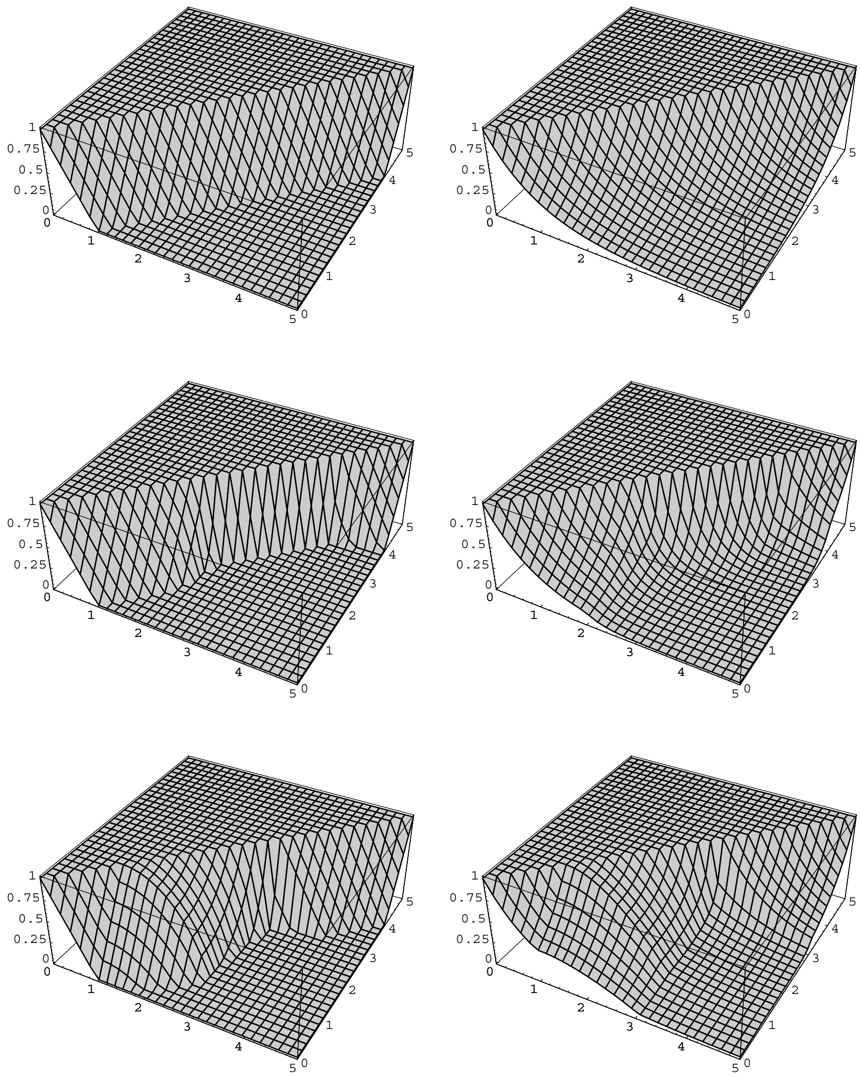
The crisp relation $\leq$ represents the well-known diamond lattice. Just by checking all possible combinations, one easily verifies that E is, indeed, a $T_{\mathbf{L}}$-equivalence. Applying the construction (4.8) yields the relation R defined as follows:

E	a	b	c	d
a	1	1	1	1
b	0.5	1	0	1
c	0.5	0	1	1
d	0	0.5	0.5	1

Considering the triple (b, a, c) shows that R is not $T_{\mathbf{L}}$-transitive:

$$T(R(b,a), R(a,c)) = T(0.5, 1) = 0.5 \not\leq 0 = R(b,c)$$

As obvious from the above example, T-transitivity becomes a crucial property if there are elements which are incomparable with respect to the



L_1 L_2

L_3 L_4

L_5 L_6

Figure 4.3: Six non-trivial examples of strongly linear fuzzy orderings.

crisp ordering. Basically, the construction (4.8) means nothing else than the $S_{\mathbf{M}}$-union of the crisp ordering $\lesssim$ and the fuzzy equivalence relation E. We have seen in 4.1 that T-transitivity cannot be guaranteed for unions of T-transitive fuzzy relations. Nevertheless, we can formulate a sufficient condition such that the construction principle (4.8) can also be applied for partial orderings.

4.30. Theorem. *Suppose $\lesssim$ to be a partial ordering on a domain X and E to be a T-equivalence which is compatible with $\lesssim$. If the property*

$$\forall x, y, z \in X: \ (x \not\lesssim z \wedge z \not\lesssim x) \Longrightarrow E(x, z) \geq \max(E(x, y), E(y, z)) \quad (4.9)$$

is additionally satisfied, the fuzzy relation

$$L(x, y) = \begin{cases} 1 & \text{if } x \lesssim y \\ E(x, y) & \text{otherwise} \end{cases} \quad (4.10)$$

defines a T-E-ordering.

Proof. Reflexivity and T-E-antisymmetry follow in the same way as in the proof of Theorem 4.27. If three elements x, y, and z are pairwise comparable, T-transitivity follows as in the proof of Theorem 4.27. As long as $x \not\lesssim y$ and $y \not\lesssim x$, T-transitivity is fulfilled because of the T-transitivity of E. Hence, T-transitivity could only be violated in one of the following cases:

$$x \lesssim y \wedge y \not\lesssim z \wedge x \not\lesssim z \wedge z \not\lesssim x \quad (4.11)$$
$$x \not\lesssim y \wedge y \lesssim z \wedge x \not\lesssim z \wedge z \not\lesssim x \quad (4.12)$$

Taking into account that condition (4.9) holds, condition (4.11) implies

$$T(L(x, y), L(y, z)) = T(1, E(y, z)) = E(y, z) \leq E(x, z) = L(x, z).$$

Analogously, condition (4.12) entails

$$T(L(x, y), L(y, z)) = T(E(x, y), 1) = E(x, y) \leq E(x, z) = L(x, z)$$

and we have proved that L is T-transitive. $\square$

The above construction is of particular importance, not only from the theoretical point of view, but also for all further investigations concerning application. Thus, it is justified to invent an own name for fuzzy orderings admitting this resolution.

4.31. Definition. A T-E-ordering L is called *direct fuzzification* of a crisp ordering $\lesssim$ if and only if it can be represented as in (4.10).

Chapter 5

From Hulls to Hedges

5.1 Motivation

We briefly mentioned in Chapter 1 that ordering-based operators, such as *'at least'*, *'at most'*, *'between'*, and so forth, could be particularly useful for applications related to fuzzy systems. The fact remains that we are still lacking a way how to compute such expressions in vague environments. In order to have a universal approach, which is applicable in a wide variety of practical problems, at least the following two properties should be satisfied:

(i) If there is a predefined notion of similarity in the given environment, the above operators should take it into account. Stressing the example of heights of persons for the very last time, this means that *'at least 180'* should not exclude 179.9 completely, because the two values are almost indistinguishable.

(ii) For using these expressions as modifiers—so-called hedges[1]—in the language of a rule-based fuzzy system, they should be applicable to fuzzy sets, because the atomic expressions are usually represented by fuzzy sets instead of crisp values.

Assume that the semantics of the connective *'and'* in a certain vague environment is specified as a t-norm T and that the underlying concept of similarity is specified as a T-equivalence E. In accordance to (i), if we want to compute truth values of expressions, such as *'y is at least as good/large/high*

[1]Some researchers use the term "adverb" in direct analogy to natural language. We will use "hedge" since it is, by far, the more common designation.

as x', for crisp values x and y, a T-E-ordering L is the perfect, ready-made choice:

$$\mu_{at\ least\ x}(y) = L(x, y) \tag{5.1}$$

Of course, this definition achieves the goal stated in criterion (i) above, because of the E-reflexivity of L. Moreover, if there is already a predefined crisp context of ordering $\lesssim$, the compatibility between E and $\lesssim$ provides a simple criterion for checking whether L respects that ordering.

In order to meet condition (ii), we have to find a way to generalize (5.1) to fuzzy subsets, for which the hull/image with respect to L is the natural candidate:

$$\mu_{at\ least\ A}(x) = \mu_{H_L(A)}(x) = \sup\{T(\mu_A(y), L(y, x)) \mid y \in X\}$$

Throughout the remaining chapter, assume that T is a left-continuous t-norm and that L is an arbitrary but fixed T-E-ordering on a domain X. Let us denote the hull operators H_L and H_E with ATL and EXT, respectively.

Before turning to more specific characterizations, we briefly discuss some properties of $\mathrm{ATL}(A)$ which hold without introducing any further restrictions.

5.1. Lemma. $\mathrm{ATL}(A)$ *is extensional for any fuzzy set* $A \in \mathcal{F}(X)$. *Furthermore, if* L *fuzzifies a crisp ordering* $\lesssim$ *then the membership function of* $\mathrm{ATL}(A)$ *is non-decreasing with respect to* $\lesssim$, *i.e.*

$$\forall x, y \in X : \ x \lesssim y \Longrightarrow \mu_{\mathrm{ATL}(A)}(x) \leq \mu_{\mathrm{ATL}(A)}(y). \tag{5.2}$$

Proof. We know that $\mathrm{ATL}(A)$ is L-congruent. Trivially, E-reflexivity of L implies that E is a subrelation of L and we obtain from Corollary 2.51 that $\mathrm{ATL}(A)$ is E-congruent, i.e. extensional according to Definition 2.61. The same argument applies to the crisp ordering $\lesssim$ which is a subrelation of L, too. Hence, $\mathrm{ATL}(A)$ is $\lesssim$-congruent

$$\forall x, y \in X : \ T\big(\mu_{\mathrm{ATL}(A)}(x), \chi_{\lesssim}(x, y)\big) \leq \mu_{\mathrm{ATL}(A)}(y),$$

which is, obviously, equivalent to the monotonicity (5.2). □

The previous lemma already gives a hint how the properties of the hull $\mathrm{ATL}(A)$ are determined by the properties of the fuzzy ordering L, in particular when it concerns interaction with a crisp ordering $\lesssim$. Now let us turn to the special case of direct fuzzifications, where much more is known about the connection between L and $\lesssim$.

5.2 Hulls with Respect to Direct Fuzzifications

We have seen in the previous chapter that direct fuzzifications play a very special role. In this section, we want to exploit in which way hulls with respect to direct fuzzifications can be characterized utilizing their particular properties.

So, throughout this section, let us assume that L is represented as a direct fuzzification of a crisp, not necessarily linear ordering $\lesssim$:

$$L(x,y) = \begin{cases} 1 & \text{if } x \lesssim y \\ E(x,y) & \text{otherwise} \end{cases} \tag{5.3}$$

The hull operator with respect to $\lesssim$ will be denoted LTR[2]:

$$\mu_{\mathrm{LTR}(A)}(x) = \sup\{T(\mu_A(y), \chi_{\lesssim}(y,x) \mid y \in X\} = \sup\{\mu_A(y) \mid y \lesssim x\}$$

Of course, for any fuzzy set A, $\mathrm{LTR}(A)$ is the smallest superset of A the membership function of which is non-decreasing.

Direct fuzzifications are nothing else than $S_{\mathbf{M}}$-unions of crisp orderings and compatible fuzzy equivalence relations. As a first interesting result, we obtain that this property, in some sense, transfers to hull operators, too.

5.2. Lemma. *Suppose that A is an arbitrary fuzzy subset of X. Then, for all $x \in X$, the representation*

$$\mu_{\mathrm{ATL}(A)}(x) = \max\big(\mu_{\mathrm{LTR}(A)}(x), \mu_{\mathrm{EXT}(A)}(x)\big)$$

holds which is equivalent to

$$\mathrm{ATL}(A) = \mathrm{LTR}(A) \cup_{S_{\mathbf{M}}} \mathrm{EXT}(A).$$

Proof. Taking the representation (5.3) into account, we obtain

$$\begin{aligned} \mu_{\mathrm{ATL}(A)}(x) &= \sup\{T(\mu_A(y), L(y,x)) \mid y \in X\} \\ &= \max\big(\sup\{T(\mu_A(y), L(y,x)) \mid y \lesssim x\}, \\ &\qquad\quad \sup\{T(\mu_A(y), L(y,x)) \mid y \not\lesssim x\}\big) \\ &= \max\big(\sup\{T(\mu_A(y), 1) \mid y \lesssim x\}, \\ &\qquad\quad \sup\{T(\mu_A(y), E(y,x)) \mid y \not\lesssim x\}\big) \\ &= \max\big(\mu_{\mathrm{LTR}(A)}(x), \sup\{T(\mu_A(y), E(y,x)) \mid y \not\lesssim x\}\big). \end{aligned}$$

[2]Short for "left-to-right continuation".

Obviously, the inequality

$$T(\mu_A(y), E(y,x)) \leq \mu_A(y)$$

holds for all $x, y \in X$. Hence,

$$\sup\{T(\mu_A(y), E(y,x)) \mid y \lesssim x\} \leq \sup\{\mu_A(y) \mid y \lesssim x\},$$

and we can deduce the following:

$$\begin{aligned}
\mu_{\mathrm{ATL}(A)}(x) &= \max\big(\mu_{\mathrm{LTR}(A)}(x), \sup\{T(\mu_A(y), E(y,x)) \mid y \not\lesssim x\}\big)\\
&= \max\big(\mu_{\mathrm{LTR}(A)}(x), \sup\{T(\mu_A(y), E(y,x)) \mid y \lesssim x\},\\
&\qquad\qquad\qquad\quad \sup\{T(\mu_A(y), E(y,x)) \mid y \not\lesssim x\}\big)\\
&= \max\big(\mu_{\mathrm{LTR}(A)}(x), \sup\{T(\mu_A(y), E(y,x)) \mid y \in X\}\big)\\
&= \max\big(\mu_{\mathrm{LTR}(A)}(x), \mu_{\mathrm{EXT}(A)}(x)\big)
\end{aligned}$$ □

With the help of Lemma 5.2, we can show that the hull operators LTR and EXT commute and that their composition, in either case, equals ATL.

5.3. Theorem. *For all fuzzy subsets $A \in \mathcal{F}(X)$, the following equalities hold:*

$$\mathrm{ATL}(A) = \mathrm{LTR}(\mathrm{EXT}(A)) = \mathrm{EXT}(\mathrm{LTR}(A)) \tag{5.4}$$

Proof. We know from Lemma 5.2 that $\mathrm{ATL}(A)$ is a superset of both $\mathrm{LTR}(A)$ and $\mathrm{EXT}(A)$ while Lemma 5.1 shows that $\mathrm{ATL}(A)$ is extensional and has a non-decreasing membership function. Directly from the extremal properties of hulls (cf. Lemma 2.54 and Corollary 2.55), we obtain the following inclusions:

$$\begin{aligned}
\mathrm{ATL}(A) &\supseteq \mathrm{LTR}(\mathrm{EXT}(A))\\
\mathrm{ATL}(A) &\supseteq \mathrm{EXT}(\mathrm{LTR}(A))
\end{aligned}$$

It remains to clarify whether we can have equalities in the above inequalities. Since hull operators with respect to reflexive fuzzy relations always yield supersets, we get

$$\begin{aligned}
\mathrm{EXT}(A) &\subseteq \mathrm{LTR}(\mathrm{EXT}(A)),\\
\mathrm{LTR}(A) &\subseteq \mathrm{EXT}(\mathrm{LTR}(A)).
\end{aligned}$$

On the other hand, the monotonicity of hull operators (cf. Corollary 2.55) entails

$$\begin{aligned}
\mathrm{LTR}(A) &\subseteq \mathrm{LTR}(\mathrm{EXT}(A)),\\
\mathrm{EXT}(A) &\subseteq \mathrm{EXT}(\mathrm{LTR}(A)).
\end{aligned}$$

$$\begin{tikzcd} A \arrow[r, "\mathrm{EXT}"] \arrow[d, "\mathrm{LTR}"'] & \mathrm{EXT}(A) \arrow[d, "\mathrm{LTR}"'] \\ \mathrm{LTR}(A) \arrow[r, "\mathrm{EXT}"] & \mathrm{ATL}(A) \end{tikzcd}$$

Figure 5.1: A commutative diagram depicting the relationship (5.4) for a given fuzzy set A

Using Lemma 5.2 and elementary properties of the maximum, we come to the following conclusions

$$\begin{aligned} \mathrm{ATL}(A) &\subseteq \mathrm{LTR}(\mathrm{EXT}(A)), \\ \mathrm{ATL}(A) &\subseteq \mathrm{EXT}(\mathrm{LTR}(A)), \end{aligned}$$

and the proof is completed. □

The correspondence (5.4) can be interpreted as a commutative diagram which is visualized in Figure 5.1.

5.4. Remark. All the present achievements of this section can easily be transferred to the inverse of L. Let us introduce the operator ATM[3] as synonym for the hull operator of the inverse fuzzy ordering L^{-1}

$$\mu_{\mathrm{ATM}(A)}(x) = \mu_{H_{L^{-1}}}(x) = \sup\{T(\mu_A(y), L(x,y)) \mid y \in X\}$$

and RTL[3] for the hull with respect to the inverse of $\lesssim$:

$$\mu_{\mathrm{RTL}(A)}(x) = \sup\{\mu_A(y) \mid y \gtrsim x\}$$

Then all the above results hold without any restriction, i.e.

$$\begin{aligned} \mathrm{ATM}(A) &= \mathrm{RTL}(A) \cup_{S_{\mathbf{M}}} \mathrm{EXT}(A) \\ &= \mathrm{RTL}(\mathrm{EXT}(A)) = \mathrm{EXT}(\mathrm{RTL}(A)). \end{aligned} \tag{5.5}$$

Since non-decreasingness with respect to the inverse of $\lesssim$ means nothing else than non-increasingness with respect to $\lesssim$, we obtain that the membership functions of $\mathrm{RTL}(A)$ and $\mathrm{ATM}(A)$ are non-increasing and that $\mathrm{RTL}(A)$ is the smallest non-increasing superset of A.

[3]ATM stands for "at most" while RTL stems from "right-to-left continuation".

As a consequence of all above results, we can prove that not only the hulls $\mathrm{EXT}(A)$, $\mathrm{LTR}(A)$, and $\mathrm{RTL}(A)$, but also $\mathrm{ATL}(A)$ and $\mathrm{ATM}(A)$ have extremal properties.

5.5. Corollary. *Suppose that A is an arbitrary fuzzy subset of X. Then* $\mathrm{ATL}(A)$ *is the smallest extensional superset of A which has a non-decreasing membership function. Analogously,* $\mathrm{ATM}(A)$ *is the smallest extensional superset of A which has a non-increasing membership function.*

Proof. We know from Lemma 5.1 that $\mathrm{ATL}(A)$ is extensional and that its membership function is non-decreasing. It remains to show that it is the smallest extensional superset fulfilling both properties. Assume that there is a $B \supseteq A$ which is extensional and has a non-decreasing membership function. Due to the minimality of hulls (cf. Lemma 2.54), B must be a superset of $\mathrm{EXT}(A)$ and $\mathrm{LTR}(A)$. However, taking Lemma 5.2 into account, these inclusions imply

$$B \supseteq \mathrm{EXT}(A) \cup_{S_{\mathbf{M}}} \mathrm{LTR}(A) = \mathrm{ATL}(A).$$

The same argument can be applied analogously to prove the corresponding minimality of $\mathrm{ATM}(A)$. □

5.3 Convex Hulls and their Characterization

Since it will be of particular importance in Chapter 6, we will now study the convexity of hulls with respect to fuzzy orderings and a way how to construct convex hulls and extensional convex hulls. Throughout this subsection, assume that X is a domain equipped with an ordering $\lesssim$. Furthermore, let us assume that L is a T-E-ordering which fuzzifies $\lesssim$, where T is a left-continuous t-norm.

5.6. Lemma. *The fuzzy sets* $\mathrm{LTR}(A)$*,* $\mathrm{RTL}(A)$*,* $\mathrm{ATL}(A)$*, and* $\mathrm{ATM}(A)$ *are convex for any $A \in \mathcal{F}(X)$.*

Proof. We already know that $\mathrm{LTR}(A)$ and $\mathrm{ATL}(A)$ have a non-decreasing membership functions while the membership functions of both $\mathrm{RTL}(A)$ and $\mathrm{ATM}(A)$ are non-increasing. Then convexity follows from Proposition 2.10. □

5.7. Definition. For a given fuzzy subset A of X, the operators CVX and ECX are defined as

$$\begin{aligned}\mathrm{CVX}(A) &= \mathrm{LTR}(A) \cap_{T_{\mathbf{M}}} \mathrm{RTL}(A)\\ \mathrm{ECX}(A) &= \mathrm{ATL}(A) \cap_{T_{\mathbf{M}}} \mathrm{ATM}(A)\end{aligned}$$

One can easily deduce from the definition of convexity and the associativity of $T_{\mathbf{M}}$, that the $T_{\mathbf{M}}$-intersection of two convex fuzzy subsets is again convex (note that this is not necessarily true for intersections with respect to other t-norms). Hence, we immediately see that $\mathrm{CVX}(A)$ and $\mathrm{ECX}(A)$ are convex fuzzy sets. The next result proves an extremal property of $\mathrm{CVX}(A)$, which justifies to call $\mathrm{CVX}(A)$ the *convex hull* of A.

5.8. Lemma. *For any fuzzy subset A of X, $\mathrm{CVX}(A)$ is the smallest convex superset.*

Proof. Assume that B is a convex superset of A. Taking an arbitrary $y \in X$, the following must hold:

$$\forall x, z \in X: \; x \lesssim y \lesssim z \Longrightarrow \mu_B(y) \geq \min(\mu_B(x), \mu_B(z))$$

Since this holds for all chains $x \lesssim y \lesssim z$, we can even take the supremum and the following is obtained:

$$\begin{aligned}
\mu_B(y) &\geq \sup\{\min(\mu_B(x), \mu_B(z)) \mid x \lesssim y \lesssim z\} \\
&= \min\big(\sup\{\mu_B(x) \mid x \lesssim y\}, \sup\{\mu_B(z) \mid y \lesssim z\}\big) \\
&= \min\big(\mu_{\mathrm{LTR}(B)}(y), \mu_{\mathrm{RTL}(B)}(y)\big) \\
&\geq \min\big(\mu_{\mathrm{LTR}(A)}(y), \mu_{\mathrm{RTL}(A)}(y)\big) \\
&= \mu_{\mathrm{CVX}(A)}(y)
\end{aligned}$$

The fuzzy set B was supposed to be an arbitrary convex superset of A. Therefore, $\mathrm{CVX}(A)$ must be the smallest convex superset of A. □

Now let us study in which way $\mathrm{ECX}(A)$ can be represented by means of convex and extensional hulls if L is a direct fuzzification of $\lesssim$. First of all, we can prove an an analogue of Lemma 5.2.

5.9. Lemma. *Provided that L is a direct fuzzification of L, the following holds for all fuzzy subsets $A \in \mathcal{F}(X)$:*

$$\mathrm{ECX}(A) = \mathrm{CVX}(A) \cup_{S_{\mathbf{M}}} \mathrm{EXT}(A) \tag{5.6}$$

Proof. Using that, for $T_{\mathbf{M}}$ and $S_{\mathbf{M}}$, the laws of distributivity hold, we obtain the following from Lemma 5.2:

$$\begin{aligned}
\mu_{\mathrm{ECX}(A)}(x) &= \min\big(\mu_{\mathrm{ATL}(A)}(x), \mu_{\mathrm{ATM}(A)}(x)\big) \\
&= \min\big(\max(\mu_{\mathrm{LTR}(A)}(x), \mu_{\mathrm{EXT}(A)}(x)), \\
&\qquad \max(\mu_{\mathrm{RTL}(A)}(x), \mu_{\mathrm{EXT}(A)}(x))\big) \\
&= \max\big(\min(\mu_{\mathrm{LTR}(A)}(x), \mu_{\mathrm{RTL}(A)}(x)), \mu_{\mathrm{EXT}(A)}(x)\big) \\
&= \max\big(\mu_{\mathrm{CVX}(A)}(x), \mu_{\mathrm{EXT}(A)}(x)\big)
\end{aligned}$$

□

Finally, we can show a representation of $\mathrm{ECX}(A)$ which can be considered as a direct analogue of Theorem 5.3. Another consequence of the following result is that it is definitely appropriate to call $\mathrm{ECX}(A)$ the *extensional convex hull* of A.

5.10. Theorem. *Suppose that L directly fuzzifies $\lesssim$. For any fuzzy subset A of X, the following equalities hold:*

$$\mathrm{ECX}(A) = \mathrm{CVX}(\mathrm{EXT}(A)) = \mathrm{EXT}(\mathrm{CVX}(A)) \tag{5.7}$$

Moreover, $\mathrm{ECX}(A)$ *is the smallest superset of A which is extensional and convex.*

Proof. Using the representations (5.4) and (5.5), we immediately obtain from the definition of $\mathrm{CVX}(A)$ that

$$\begin{aligned}\mathrm{ECX}(A) &= \mathrm{ATL}(A) \cap_{T_{\mathbf{M}}} \mathrm{ATM}(A)\\ &= \mathrm{LTR}(\mathrm{EXT}(A)) \cap_{T_{\mathbf{M}}} \mathrm{RTL}(\mathrm{EXT}(A))\\ &= \mathrm{CVX}(\mathrm{EXT}(A)).\end{aligned}$$

We already know that $\mathrm{ECX}(A)$ is convex. On the other hand, $\mathrm{ECX}(A)$ is the $T_{\mathbf{M}}$-intersection of two extensional fuzzy sets (cf. Lemma 5.1). Thus, Lemma 2.52 immediately implies that $\mathrm{ECX}(A)$ must be extensional. With the minimality of hulls, we obtain

$$\mathrm{ECX}(A) \supseteq \mathrm{EXT}(\mathrm{CVX}(A)).$$

Since hull operators are monotonic with respect to inclusion (cf. Corollary 2.55) and always yield supersets, we get

$$\begin{aligned}\mathrm{EXT}(\mathrm{CVX}(A)) &\supseteq \mathrm{EXT}(A),\\ \mathrm{EXT}(\mathrm{CVX}(A)) &\supseteq \mathrm{CVX}(A).\end{aligned}$$

Then Lemma 5.9 entails

$$\mathrm{ECX}(A) \subseteq \mathrm{EXT}(\mathrm{CVX}(A))$$

which proves (5.7).

Now assume that B is an extensional and convex superset of A. Since extensionality implies $B \supseteq \mathrm{EXT}(A)$ while convexity implies $B \supseteq \mathrm{CVX}(A)$, we see that

$$B \supseteq \mathrm{CVX}(A) \cup_{S_{\mathbf{M}}} \mathrm{EXT}(A) = \mathrm{ECX}(A)$$

and the minimality of $\mathrm{ECX}(A)$ is proved as well. $\square$

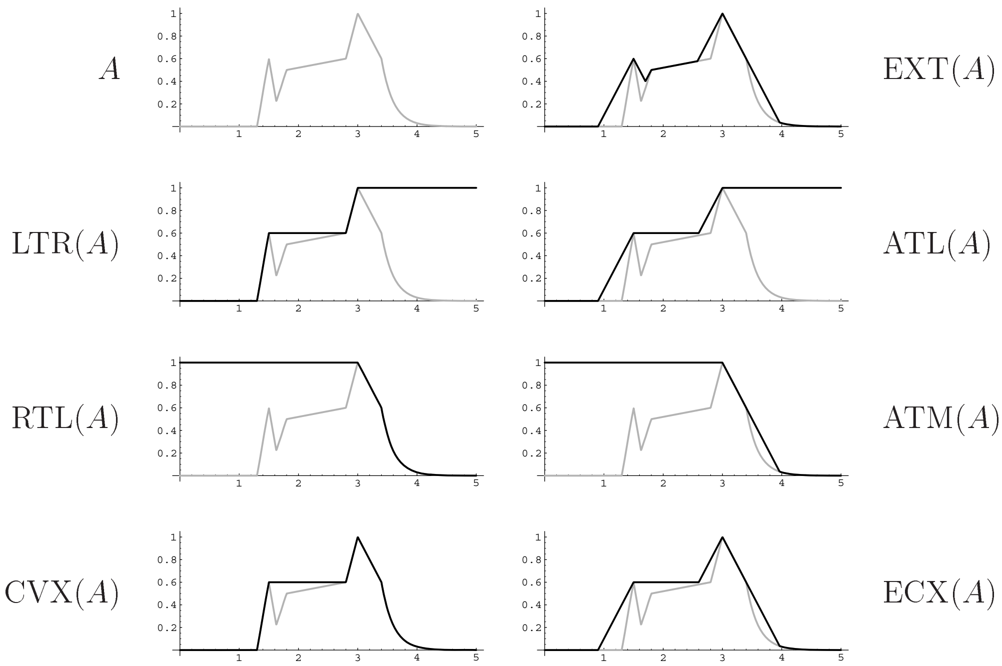


Figure 5.2: A fuzzy set $A \in \mathcal{F}(\mathbb{R})$ and the results which are obtained when applying various ordering-related hull operators.

5.11. Example. Hull operators are rather abstract objects. In order to gain more insight into the principles of the results we have discovered so far, let us consider an example of a fuzzy subset of $X = \mathbb{R}$, where we use the natural ordering of real numbers $\leq$. We assume $T = T_{\mathbf{L}}$ and the two fuzzy relations E and L to be defined as follows:

$$E(x, y) = \max(1 - |x - y|, 0)$$

$$L(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ \max(1 - x + y, 0) & \text{otherwise} \end{cases}$$

Figure 5.2 shows a non-trivial fuzzy subset of $\mathbb{R}$ and the results which are obtained when applying various hull operators.

5.4 The Role of the Extension Principle

Now we want to examine in which way the hulls we have discussed so far can be represented and interpreted by means of the extension principle.

First of all, it is worth to mention that the definitions of the hulls LTR, RTL, and CVX also include the case of crisp sets, where, obviously, the left-to-right continuation of a crisp set M is represented as

$$\mathrm{LTR}(M) = \{y \in X \mid \exists x \in M : \; x \lesssim y\}.$$

The same holds analogously for RTL:

$$\mathrm{RTL}(M) = \{y \in X \mid \exists x \in M : \; x \gtrsim y\}$$

It is trivial to see that the following holds for crisp sets M:

$$\mathrm{CVX}(M) = \mathrm{LTR}(M) \cap \mathrm{RTL}(M)$$

As a consequence, one easily verifies that $\mathrm{CVX}(M)$ can be represented in the following way:

$$\mathrm{CVX}(M) = \{y \in X \mid \exists x, z \in M : \; x \lesssim y \lesssim z\}$$

Proposition 2.9 shows that convexity is uniquely characterized by the connectedness of all α-cuts. It is, therefore, not so surprising that the hulls can also be characterized by means of the extension principle.

5.12. Theorem. *For any fuzzy subset $A \in \mathcal{F}(X)$, the following holds for all $\alpha \in [0, 1)$:*

$$\begin{aligned}
[\mathrm{LTR}(A)]_{\underline{\alpha}} &= \mathrm{LTR}\big([A]_{\underline{\alpha}}\big)\\
[\mathrm{RTL}(A)]_{\underline{\alpha}} &= \mathrm{RTL}\big([A]_{\underline{\alpha}}\big)\\
[\mathrm{CVX}(A)]_{\underline{\alpha}} &= \mathrm{CVX}\big([A]_{\underline{\alpha}}\big)
\end{aligned}$$

Proof. Let $\alpha \in [0, 1)$ and $x \in X$ be arbitrary but fixed. Then the following equivalences hold:

$$\begin{aligned}
x \in [\mathrm{LTR}(A)]_{\underline{\alpha}} &\iff \sup\{\mu_A(y) \mid y \lesssim x\} > \alpha\\
&\iff \exists y \in [A]_{\underline{\alpha}} : \; y \lesssim x\\
&\iff x \in \mathrm{LTR}\big([A]_{\underline{\alpha}}\big)
\end{aligned}$$

The same technique can be applied to prove the second identity.

One easily verifies that the following equality is true for all fuzzy subsets $A, B \in \mathcal{F}(X)$ and all $\alpha \in [0, 1)$:

$$[A \cap_{T_{\mathbf{M}}} B]_{\underline{\alpha}} = [A]_{\underline{\alpha}} \cap [B]_{\underline{\alpha}}$$

Then we can prove the third assertion as consequence of the first and the second one:

$$\begin{aligned}[\mathrm{CVX}(A)]_{\underline{\alpha}} &= [\mathrm{LTR}(A) \cap_{T_{\mathbf{M}}} \mathrm{RTL}(A)]_{\underline{\alpha}} \\ &= [\mathrm{LTR}(A)]_{\underline{\alpha}} \cap [\mathrm{RTL}(A)]_{\underline{\alpha}} \\ &= \mathrm{LTR}\big([A]_{\underline{\alpha}}\big) \cap \mathrm{RTL}\big([A]_{\underline{\alpha}}\big) \\ &= \mathrm{CVX}\big([A]_{\underline{\alpha}}\big)\end{aligned}$$

□

Note that there is no analogue for non-strict α-cuts. In order to see this, we consider the following example with $X = \mathbb{R}$:

$$\mu_A(x) = \begin{cases} x & \text{if } x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$

The left-to-right continuation of A is, obviously, given as

$$\mu_{\mathrm{LTR}(A)}(x) = \begin{cases} x & \text{if } x \in (0,1), \\ 1 & \text{if } x \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

and we obtain $[\mathrm{LTR}(A)]_1 = [1, \infty)$ but $\mathrm{LTR}([A]_1) = \mathrm{LTR}(\emptyset) = \emptyset$.

Moreover, the assertions of Theorem 5.12 do not hold if we introduce indistinguishability, i.e. if we replace LTR by ATL, RTL by ATM, and CVX by ECX—just consider that α-cuts of hulls are always crisp sets while the hulls of α-cuts are, in most cases, non-trivial fuzzy sets.

5.5 More about Ordering-Based Hedges

The previous discussions enable us to define some other ordering-based operators which could be useful in applications. In the following, assume that L is a T-E-ordering on the domain X and that N is an involution.

5.13. Definition. Let A be an arbitrary fuzzy subset of X. Then we can define the following unary modifiers:

1. ‘Strictly greater than A’ ($\mathrm{SGT}(A)$):

$$\mathrm{SGT}(A) = \mathrm{ATL}(A) \cap_{T_{\mathbf{M}}} \complement_N \mathrm{ECX}(A)$$

2. ‘Strictly less than A’ ($\mathrm{SLS}(A)$):

$$\mathrm{SLS}(A) = \mathrm{ATM}(A) \cap_{T_{\mathbf{M}}} \complement_N \mathrm{ECX}(A)$$

3. ‘Within A’ (WIT(A)):

$$\mathrm{WIT}(A) = \mathrm{ECX}(A) \cap_{T_{\mathbf{M}}} \complement_N \mathrm{EXT}(A)$$

Note that SGT(A) does not necessarily coincide with the hull of A with respect to the dual relation L^d. The same applies to SLS(A) and the complement relation of L. The operator WIT provides a method for extracting “holes” in non-convex fuzzy sets, where, obviously, WIT(A) is empty if A is convex.

5.14. Definition. For two fuzzy subsets $A, B \in \mathcal{F}(X)$, we can define the following two binary modifiers:

1. ‘Extensional convex closure of A and B’ (ECL(A, B)):

$$\mathrm{ECL}(A, B) = \mathrm{ECX}(A \cup_{S_{\mathbf{M}}} B)$$

2. ‘Between A and B’ (BTW(A, B)):

$$\mathrm{BTW}(A, B) = \big(\mathrm{SGT}(A) \cap_{T_{\mathbf{M}}} \mathrm{SLS}(B)\big) \cup_{S_{\mathbf{M}}} \big(\mathrm{SGT}(B) \cap_{T_{\mathbf{M}}} \mathrm{SLS}(A)\big)$$

Note that these two operator can also be used for grouping neighboring rules with the objective of size reduction. Moreover, the latter can be used to fill the gap between two distant non-overlapping fuzzy sets—a fundamental task in rule interpolation [38, 39].

Chapter 6

Orderings of Fuzzy Sets

6.1 Motivation

There is no doubt that orderings and rankings play a central role in any discipline which is considered to be related to decision making. Admitting vagueness or impreciseness naturally results in the need for specifying vague preferences in crisp domains, but also in the demand for a framework in which it is even possible to decide between fuzzy alternatives. Thus, it is not surprising that orderings and rankings of fuzzy sets have become main objects of study in fuzzy decision analysis. Albeit only scarcely recognized, orderings of fuzzy sets could also be integrated fruitfully in areas related to fuzzy systems and fuzzy control:

1. We briefly mentioned in Chapter 1 that it is a typical method in the design of fuzzy systems and controllers to decompose the domains of the various system variables into a certain number of fuzzy subsets by means of the orderings of these domains. Reversely, the ordering of the fuzzy subsets, which define the actual meaning of linguistic expressions, is an important point when it concerns interpretability.

2. As any interpolation technique requires a way to select the data between which the interpolation should take place, rule interpolation does not make a significant difference. More specifically, if an observation does not match any antecedent in the rule base, one has to select two or more rules for interpolation. It seems natural to take rules such that the observation is lying between the fuzzy sets specified in the antecedents of these rules.

Since the second half of the 1970s, a host of methods for ordering or ranking fuzzy sets has been published (see [7, 66, 68, 69] for detailed reviews). In order to find profound motivations for adding yet another approach, let us consider some common characteristics of the existing methods:

1. As long as linguistic expressions are represented by fuzzy subsets of numerical domains, there is a certain context-dependent notion of indistinguishability. It could be desirable to take this indistinguishability into account, since not only the ranking of alternatives itself, but also the information, that the difference between two alternatives is, more or less, neglectable, could be of interest. All existing methods, however, do not offer the opportunity to integrate indistinguishability which often leads to undesired, counter-intuitive preciseness.

2. All methods are defined for so-called *fuzzy quantities*—fuzzy subsets of the real line $\mathbb{R}$. Not only from the theoretical but also from the practical point of view, it could be interesting to consider arbitrary ordered domains, without any restriction concerning analytical properties, cardinality, or linearity of the ordering.

3. The applicability of many ordering methods is restricted to fuzzy quantities having special properties, such as convexity, normality, or continuity. The motivation for such restrictions is to guarantee some desirable properties, for example, antisymmetry.

The purpose of this chapter is to define and investigate an ordering method for arbitrary fuzzy subsets of an arbitrary ordered domain where indistinguishability is taken into account, too.

6.2 A Novel Approach based on Fuzzy Orderings

With the objective in mind that the new ordering procedure should be represented as a binary relation on $\mathcal{F}(X)$ which is at least a preordering, i.e. reflexive and transitive, let us first consider the simple case of real intervals, where an often-used ordering procedure is given as follows:

$$[a, b] \leq_I [c, d] \iff a \leq c \land b \leq d$$

It is easy to check that $\leq_I$ is an ordering. The inequality $a \leq c$ means that there are no parts of $[c, d]$ which are below the entire interval $[a, b]$. The inequality $b \leq d$, analogously, means that there are no elements of $[a, b]$ which lie completely above $[c, d]$. This criterion can be generalized to arbitrary crisp subsets of an ordered set $(X, \lesssim)$ as follows:

$$\begin{aligned} M_1 \lesssim_I M_2 \iff & \big(\forall x \in M_2\ \exists y \in M_1 :\ y \lesssim x\big) \wedge \\ & \big(\forall x \in M_1\ \exists y \in M_2 :\ x \lesssim y\big) \end{aligned} \tag{6.1}$$

The following lemma provides an equivalent formulation by means of hulls which will be the basis of all our further generalizations.

6.1. Lemma. *Let a domain X be equipped with an ordering $\lesssim$. With the above notations, the following holds for all $M_1, M_2 \subseteq X$:*

$$\begin{aligned} \mathrm{LTR}(M_1) \supseteq \mathrm{LTR}(M_2) &\iff \forall x \in M_2\ \exists y \in M_1 :\ y \lesssim x \\ \mathrm{RTL}(M_1) \subseteq \mathrm{RTL}(M_2) &\iff \forall x \in M_1\ \exists y \in M_2 :\ x \lesssim y \end{aligned}$$

As an immediate consequence, we obtain

$$M_1 \lesssim_I M_2 \iff \big(\mathrm{LTR}(M_1) \supseteq \mathrm{LTR}(M_2) \wedge \mathrm{RTL}(M_1) \subseteq \mathrm{RTL}(M_2)\big). \tag{6.2}$$

Proof. It is easy to see from the definition

$$\mathrm{LTR}(M_1) = \{x \in X \mid \exists y \in M_1 :\ y \lesssim x\}$$

that $M_2 \subseteq \mathrm{LTR}(M_1)$ is equivalent to

$$\forall x \in M_2\ \exists y \in M_1 :\ y \lesssim x.$$

Moreover, we can deduce from the minimality of hulls and the trivial inclusion $M_2 \subseteq \mathrm{LTR}(M_2)$ that the following holds:

$$M_2 \subseteq \mathrm{LTR}(M_1) \iff \mathrm{LTR}(M_2) \subseteq \mathrm{LTR}(M_1)$$

So, we have proved the following equivalence:

$$\forall x \in M_2\ \exists y \in M_1 :\ y \lesssim x \iff \mathrm{LTR}(M_1) \supseteq \mathrm{LTR}(M_2)$$

Applying analogous arguments, one can show

$$\forall x \in M_1\ \exists y \in M_2 :\ x \lesssim y \iff \mathrm{RTL}(M_1) \subseteq \mathrm{RTL}(M_2)\big).$$

These two equivalences, of course, imply (6.2). □

The equivalence (6.2) immediately implies that $\lesssim_I$ is a reflexive and transitive relation on $\mathcal{P}(X)$. Furthermore, it already gives a clear hint how to generalize $\lesssim_I$ to $\mathcal{F}(X)$:

$$A \lesssim_I B \iff \big(\mathrm{LTR}(A) \supseteq \mathrm{LTR}(B) \wedge \mathrm{RTL}(A) \subseteq \mathrm{RTL}(B)\big) \tag{6.3}$$

The only question still to be clarified is how to integrate a predefined concept of indistinguishability which is specified as a fuzzy equivalence relation E. The answer is simple—just by choosing a T-E-ordering L, where we have to assume T to be left-continuous, and replacing the operators LTR and RTL by ATL and ATM, respectively.

6.2. Definition. Let L be a fuzzy ordering on X. Then the relation $\lesssim_L$ on $\mathcal{F}(X)$ is defined in the following way:

$$A \lesssim_L B \iff \big(\mathrm{ATL}(A) \supseteq \mathrm{ATL}(B) \wedge \mathrm{ATM}(A) \subseteq \mathrm{ATM}(B)\big) \tag{6.4}$$

In the following, we will use $\lesssim_I$ and $\lesssim_L$ independently, although $\lesssim_I$ is, obviously, a special case of $\lesssim_L$. Unless stated otherwise, the fuzzy ordering L is neither supposed to be strongly linear nor to have any connection with the crisp ordering $\lesssim$. Let us, anyway, use the term "extensional convex hull" for any ECX just for convenience.

6.2.1 Basic Properties

Trivially, the relation $\lesssim_L$ is reflexive and transitive. The next result characterizes antisymmetry, or better non-antisymmetry, in a unique way.

6.3. Theorem. *With the above settings, the following holds for all fuzzy subsets $A, B \in \mathcal{F}(X)$:*

$$A \lesssim_L B \wedge A \gtrsim_L B \iff \mathrm{ECX}(A) = \mathrm{ECX}(B)$$

Proof. The above assertion is, of course, equivalent to

$$\mathrm{ATL}(A) = \mathrm{ATL}(B) \wedge \mathrm{ATM}(A) = \mathrm{ATM}(B) \iff \mathrm{ECX}(A) = \mathrm{ECX}(B).$$

The implication from the left to the right follows trivially from the definition of the ECX operator. In order to prove the non-trivial implications, consider the monotonicity of hull operators (cf. Corollary 2.55):

$$A \subseteq \mathrm{ECX}(A) \implies \mathrm{ATL}(A) \subseteq \mathrm{ATL}(\mathrm{ECX}(A))$$

Idempotency (see also Corollary 2.55), on the other hand, entails

$$\mathrm{ECX}(A) \subseteq \mathrm{ATL}(A) \implies \mathrm{ATL}(\mathrm{ECX}(A)) \subseteq \mathrm{ATL}(\mathrm{ATL}(A)) = \mathrm{ATL}(A)$$

and we have proved that

$$\mathrm{ATL}(\mathrm{ECX}(A)) = \mathrm{ATL}(A).$$

Analogously, it is possible to show

$$\mathrm{ATM}(\mathrm{ECX}(A)) = \mathrm{ATM}(A),$$

and we obtain the non-trivial implication

$$\mathrm{ECX}(A) = \mathrm{ECX}(B) \implies \mathrm{ATL}(A) = \mathrm{ATL}(B) \wedge \mathrm{ATM}(A) = \mathrm{ATM}(B)$$

which completes the proof. □

Since every crisp ordering is, of course, a fuzzy ordering, we can transfer the characterization of non-antisymmetry to the relation $\lesssim_I$.

6.4. Corollary. *The following holds for all fuzzy subsets A, B of an ordered domain $(X, \lesssim)$:*

$$A \lesssim_I B \wedge A \gtrsim_I B \iff \mathrm{CVX}(A) = \mathrm{CVX}(B)$$

In any case, we see that neither $\lesssim_I$ nor $\lesssim_L$ is guaranteed to be antisymmetric. However, we have found equivalence relations uniquely describing non-antisymmetry. Of course, we can obtain orderings by factorization with respect to the symmetric kernels of $\lesssim_I$ and $\lesssim_L$, respectively. From Theorem 6.3 and Corollary 6.4, we know that the symmetric kernels can be represented as follows:

$$\begin{aligned} A \cong_I B &\iff \mathrm{CVX}(A) = \mathrm{CVX}(B) \\ A \cong_L B &\iff \mathrm{ECX}(A) = \mathrm{ECX}(B) \end{aligned}$$

6.5. Proposition. *The relation $\lesssim_I$ is an ordering on $\mathcal{F}(X)_{/\cong_I}$ which is isomorphic to the set of convex fuzzy subsets:*

$$\mathcal{F}_I(X) = \{A \in \mathcal{F}(X) \mid A = \mathrm{CVX}(A)\}$$

Analogously, the relation $\lesssim_L$ is an ordering on $\mathcal{F}(X)_{/\cong_L}$ which is isomorphic to the set of extensional convex fuzzy subsets:

$$\mathcal{F}_L(X) = \{A \in \mathcal{F}(X) \mid A = \mathrm{ECX}(A)\}$$

Proof. According to the discussions in Section 4.3, the relations $\lesssim_I$ and $\lesssim_L$ must be orderings on $\mathcal{F}(X)_{/\cong_I}$ and $\mathcal{F}(X)_{/\cong_L}$, respectively. Moreover, the mappings

$$\begin{aligned} \mathbf{1}_I :\ \mathcal{F}(X)_{/\cong_I} &\longrightarrow \mathcal{F}_I(X) \\ \langle A \rangle &\longmapsto \mathrm{CVX}(A) \end{aligned}$$

$$\begin{aligned} \mathbf{1}_L :\ \mathcal{F}(X)_{/\cong_L} &\longrightarrow \mathcal{F}_L(X) \\ \langle A \rangle &\longmapsto \mathrm{ECX}(A) \end{aligned}$$

are easily shown to be bijective and order-preserving. □

The above results have a different quality if we compare them with the existing approaches which restricted to some special classes of fuzzy subsets from the beginning just to preserve properties, such as antisymmetry. The new method is not restricted to (extensional) convex fuzzy sets. It can distinguish between any two arbitrary fuzzy subsets as long as their (extensional) convex hulls do not coincide. Since non-antisymmetry is characterized by an equivalence relation, it could be possible to define orderings of the equivalence classes in order to obtain a broader class of fuzzy subsets for which antisymmetry is satisfied (see 6.3.3).

6.2.2 Connections to the Extension Principle

In this subsection, let us examine which connections between the preordering $\lesssim_I$ and known approaches, which are based on the extension principle, can be established. As stated in Remark 2.56, there is a close connection between hull operations and the extension principle. The next lemma shows that the preordering $\lesssim_I$ even has a direct interpretation employing the extension principle.

6.6. Lemma. *The following holds for any two fuzzy sets $A, B \in \mathcal{F}(X)$:*

$$A \lesssim_I B \iff \forall \alpha \in [0,1) :\ [A]_{\underline{\alpha}} \lesssim_I [B]_{\underline{\alpha}}$$

Proof. First, consider Lemma 2.6:

$$\begin{aligned} A \lesssim_I B &\iff \big(\mathrm{LTR}(A) \supseteq \mathrm{LTR}(B) \wedge \mathrm{RTL}(A) \subseteq \mathrm{RTL}(B)\big) \\ &\iff \forall \alpha \in [0,1) :\ [\mathrm{LTR}(A)]_{\underline{\alpha}} \supseteq [\mathrm{LTR}(B)]_{\underline{\alpha}} \wedge \\ &\qquad\qquad\qquad\quad [\mathrm{RTL}(A)]_{\underline{\alpha}} \subseteq [\mathrm{RTL}(B)]_{\underline{\alpha}} \end{aligned}$$

Due to Theorem 5.12, this is equivalent to

$$\forall \alpha \in [0,1) :\ \mathrm{LTR}\big([A]_{\underline{\alpha}}\big) \supseteq \mathrm{LTR}\big([B]_{\underline{\alpha}}\big) \wedge \mathrm{RTL}\big([A]_{\underline{\alpha}}\big) \subseteq \mathrm{RTL}\big([B]_{\underline{\alpha}}\big),$$

which is nothing else than

$$\forall \alpha \in [0,1): \; [A]_{\underline{\alpha}} \lesssim_I [B]_{\underline{\alpha}}$$

and the proof is completed. □

As an immediate consequence, we can even prove an extension-principle-based interpretation of the general preordering $\lesssim_L$ in the case that L is a direct fuzzification.

6.7. Corollary. *Provided that a fuzzy ordering L is a direct fuzzification of $\lesssim$, the following correspondence is true for all fuzzy subsets A and B:*

$$A \lesssim_L B \iff \forall \alpha \in [0,1): \; [\mathrm{EXT}(A)]_{\underline{\alpha}} \lesssim_I [\mathrm{EXT}(B)]_{\underline{\alpha}}$$

Proof. Follows directly from Lemma 6.6 and the equalities

$$\begin{aligned} \mathrm{ATL}(A) &= \mathrm{LTR}(\mathrm{EXT}(A)) & \qquad \mathrm{ATL}(B) &= \mathrm{LTR}(\mathrm{EXT}(B)) \\ \mathrm{ATM}(A) &= \mathrm{RTL}(\mathrm{EXT}(A)) & \qquad \mathrm{ATM}(B) &= \mathrm{RTL}(\mathrm{EXT}(B)) \end{aligned}$$

(cf. Theorem 5.3 and Remark 5.4). □

The idea of employing the extension principle for extending orderings of intervals to orderings of fuzzy quantities is definitely not new. A common technique is to extend the interval ordering $\leq_I$ to convex fuzzy quantities. Lemma 6.6 implies that, if $\leq_I$ is applied to each strict α-cut [32], this is equivalent to the preordering $\lesssim_I$. The same is true for non-strict α-cuts [38] under some additional restrictions concerning continuity of the membership functions.

From this point of view, the preordering $\lesssim_I$ coincides with previously known approaches if convex fuzzy quantities are considered. The novelties, however, consist of the following points:

1. The applicability of the preorderings $\lesssim_I$ and $\lesssim_L$ is neither restricted to fuzzy quantities nor to convex fuzzy sets. Moreover, the underlying orderings are not required to be linear.

2. Unlike all existing methods, it is possible to integrate indistinguishability in the preordering $\lesssim_L$.

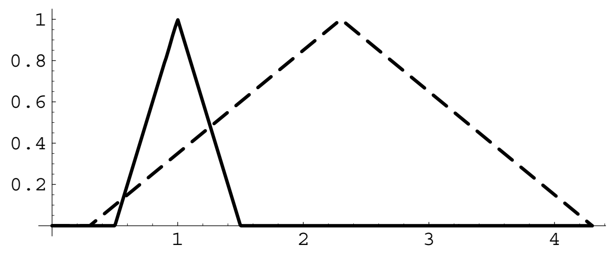


Figure 6.1: Two convex fuzzy quantities which are incomparable.

6.2.3 Weaknesses

Our initial objective was to define (pre)orderings of arbitrary fuzzy subsets of a domain for which a crisp or a fuzzy ordering is known. Now we should examine in detail whether this goal has really been achieved.

Formally, if a crisp ordering $\lesssim$ is known for an arbitrary domain X, one can easily construct the relation $\lesssim_I$. In the same way, it is possible to define the relation $\lesssim_L$ by means of hulls with respect to some fuzzy ordering L. We have found out that both relations are preorderings which violate antisymmetry only in the case that the (extensional) convex hulls of two fuzzy alternatives coincide. So, from the barely formal point of view, our initial requirements have been met. There are, however, still some weaknesses that are worth to be pointed out.

The first remark concerns the way of comparing itself. Consider the two convex fuzzy quantities in Figure 6.1. It is easy to see that, if we construct $\lesssim_I$ by means of the natural ordering of real numbers, these two triangular fuzzy quantities are incomparable. If a strongly linear fuzzy ordering L is chosen, which fuzzifies $\leq$, such that the two fuzzy quantities are extensional with respect to the underlying fuzzy equivalence relation, the situation cannot be better (note that, in this case, $\lesssim_L$ coincides with $\lesssim_I$). The question is whether it is really natural to compare vague phenomena crisply or if, as the example in Figure 6.1 suggests, this directly leads to artificial preciseness. In 6.3.1, we will present a way to overcome these difficulties.

Unfortunately, incomparability is also obtained if two fuzzy alternatives have different heights.

6.8. Lemma. *Consider a fuzzy ordering L. Then the following equalities hold for any fuzzy subset $A \in \mathcal{F}(X)$:*

$$\mathrm{height}(A) = \mathrm{height}(\mathrm{ATL}(A)) = \mathrm{height}(\mathrm{ATM}(A)) = \mathrm{height}(\mathrm{ECX}(A))$$

Proof. Since the inequality

$$T(\mu_A(y), L(y,x)) \leq \mu_A(y)$$

holds trivially for all $x, y \in X$, we obtain the following:

$$\begin{aligned}
\text{height}(\text{ATL}(A)) &= \sup_{x\in X} \mu_{\text{ATL}(A)}(x) \\
&= \sup_{x\in X} \sup_{y\in X} T(\mu_A(y), L(y,x)) \\
&\leq \sup_{y\in X} \mu_A(y) \\
&= \text{height}(A)
\end{aligned}$$

On the other hand, ATL(A) is a superset of A which implies that we must have equalities. The same argument applies to ATM(A) analogously. The fuzzy set ECX(A) is a subset of ATL(A) and ATM(A). Hence, it cannot exceed their heights. ECX(A) is, however, again a superset of A, which implies finally that

$$\text{height}(A) = \text{height}(\text{ECX}(A)). \qquad \square$$

Moreover, we immediately obtain analogous equalities if L coincides with a crisp ordering $\lesssim$:

$$\text{height}(A) = \text{height}(\text{LTR}(A)) = \text{height}(\text{RTL}(A)) = \text{height}(\text{CVX}(A))$$

It might be clear that a fuzzy set A can never be a subset of another B if height(A) exceeds height(B). We, therefore, obtain that any two fuzzy sets with different heights must be incomparable, regardless which crisp or fuzzy ordering has been chosen to define $\lesssim_L$. We will discuss this aspect in 6.3.2.

The third and last point of critique refers to (non-)antisymmetry. While Theorem 6.3 and Corollary 6.4 state that non-antisymmetry is obtained just in the case that the (extensional) convex hulls of the two alternatives coincide, Proposition 6.5 shows that fully antisymmetric orderings are obtained if we restrict ourselves to (extensional) convex fuzzy sets. One might observe, however, that these results are still not exhaustive. Consider, for instance, the two fuzzy quantities shown in Figure 6.2. Obviously, they have equal convex hulls. Hence, the relation $\lesssim_I$ cannot distinguish between them. The same happens with $\lesssim_L$ if L is chosen as a fuzzification of the canonical ordering of real numbers $\leq$. Nevertheless, one would intuitively assume that the lower set should be ranked higher than the upper one, simply looking at the positions of the "holes". In 6.3.3, we will discuss ways to refine the orderings $\lesssim_I$ and $\lesssim_L$ by defining (pre)orderings of all classes containing fuzzy sets with equal (extensional) convex hulls.

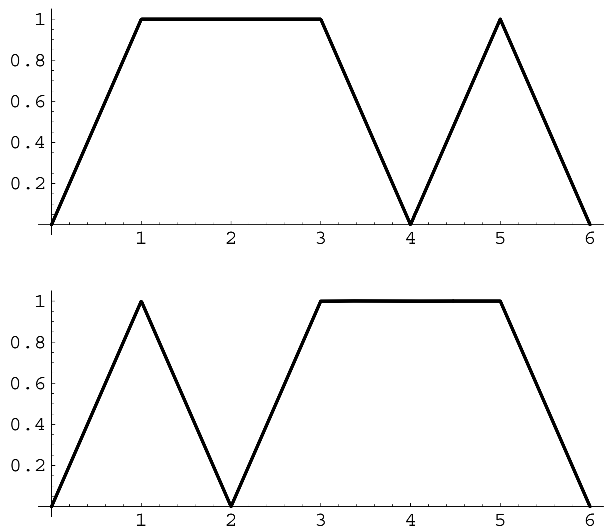


Figure 6.2: Two non-convex fuzzy quantities with equal convex hulls.

6.3 Generalizations and Extensions

The purpose of this section is to find generalizations of the two ordering methods $\lesssim_I$ and $\lesssim_L$ such that the difficulties stated in 6.2.3 are, at least partially, resolved.

6.3.1 Fuzzification

Figure 6.1 has demonstrated that crisp comparisons sometimes lead to artificial preciseness. Now we want to discuss a way for overcoming this problem by allowing intermediate degrees of inclusion—with the aim to obtain a fuzzy ordering of fuzzy sets. Throughout this subsection, let L be a fuzzy ordering with respect to some fuzzy equivalence relation E and a left-continuous t-norm T. As known from above, a fuzzy ordering L induces a crisp preordering of fuzzy subsets of X:

$$A \lesssim_L B \iff \mathrm{ATL}(A) \supseteq \mathrm{ATL}(B) \wedge \mathrm{ATM}(A) \subseteq \mathrm{ATM}(B)$$

In order to make this expression fuzzy, we need to find fuzzifications of the inclusions and the Boolean logical conjunction $\wedge$. Apparently, the relation INCL_T—its existence is guaranteed because T is supposed to be left-continuous—is *the* natural candidate as a gradual concept of inclusion while

$\wedge$ should, clearly, be replaced by some t-norm $\tilde{T}$:

$$\begin{aligned} \mathcal{L}_{\tilde{T},L}(A,B) = \tilde{T}\big(&\mathrm{INCL}_T(\mathrm{ATL}(B),\mathrm{ATL}(A)), \\ &\mathrm{INCL}_T(\mathrm{ATM}(A),\mathrm{ATM}(B))\big) \end{aligned} \tag{6.5}$$

6.9. Theorem. *If $\tilde{T}$ dominates T, the binary fuzzy relation $\mathcal{L}_{\tilde{T},L}$ is a T-$\mathcal{E}_{\tilde{T},L}$ ordering on $\mathcal{F}(X)$ with*

$$\mathcal{E}_{\tilde{T},L}(A,B) = \tilde{T}\big(\mathrm{SIM}_T(\mathrm{ATL}(A),\mathrm{ATL}(B)),\mathrm{SIM}_T(\mathrm{ATM}(A),\mathrm{ATM}(B))\big).$$

Moreover, the following holds for all $A,B \in \mathcal{F}(X)$:

$$\mathcal{E}_{\tilde{T},L}(A,B) \leq \mathrm{SIM}_T\big(\mathrm{ECX}(A),\mathrm{ECX}(B)\big) \tag{6.6}$$

Proof. Consider the two relations

$$\begin{aligned} \mathcal{L}'(A,B) &= \mathrm{INCL}_T(\mathrm{ATL}(B),\mathrm{ATL}(A)), \\ \mathcal{L}''(A,B) &= \mathrm{INCL}_T(\mathrm{ATM}(A),\mathrm{ATM}(B)). \end{aligned}$$

Obviously, both are reflexive and T-transitive (cf. Lemma 3.3). Due to Theorem 3.11, therefore, they are fuzzy orderings with respect to T and their symmetric kernels (compare with Example 3.14)

$$\begin{aligned} \mathcal{E}'(A,B) &= \min\big(\mathrm{INCL}_T(\mathrm{ATL}(B),\mathrm{ATL}(A)),\mathrm{INCL}_T(\mathrm{ATL}(A),\mathrm{ATL}(B))\big) \\ &= \mathrm{SIM}_T\big(\mathrm{ATL}(A),\mathrm{ATL}(B)\big) \\ \mathcal{E}''(A,B) &= \min\big(\mathrm{INCL}_T(\mathrm{ATM}(A),\mathrm{ATM}(B)),\mathrm{INCL}_T(\mathrm{ATM}(B),\mathrm{ATM}(A))\big) \\ &= \mathrm{SIM}_T\big(\mathrm{ATM}(A),\mathrm{ATM}(B)\big) \end{aligned}$$

and T. We can apply Theorem 4.1 and obtain that

$$\mathcal{L}_{\tilde{T},L}(A,B) = \tilde{T}\big(\mathcal{L}'(A,B),\mathcal{L}''(A,B)\big)$$

is a T-$\mathcal{E}_{\tilde{T},L}$-ordering with

$$\begin{aligned} \mathcal{E}_{\tilde{T},L}(A,B) &= \tilde{T}\big(\mathcal{E}'(A,B),\mathcal{E}''(A,B)\big) \\ &= \tilde{T}\big(\mathrm{SIM}_T(\mathrm{ATL}(A),\mathrm{ATL}(B)),\mathrm{SIM}_T(\mathrm{ATM}(A),\mathrm{ATM}(B))\big). \end{aligned}$$

Now it remains to show that the inequality (6.6) holds. Using basic properties of minimum and supremum, together with Lemma 2.37, Point 5., we obtain

$$\begin{aligned}
\mathcal{E}_{\tilde{T},L}(A,B) &= \tilde{T}\big(\mathrm{SIM}_T(\mathrm{ATL}(A),\mathrm{ATL}(B)),\mathrm{SIM}_T(\mathrm{ATM}(A),\mathrm{ATM}(B))\big)\\
&\leq \min\big(\mathrm{SIM}_T(\mathrm{ATL}(A),\mathrm{ATL}(B)),\mathrm{SIM}_T(\mathrm{ATM}(A),\mathrm{ATM}(B))\big)\\
&= \min\big(\inf_{x\in X}\overset{\leftrightarrow}{T}(\mu_{\mathrm{ATL}(A)}(x),\mu_{\mathrm{ATL}(B)}(x)),\\
&\qquad\quad \inf_{x\in X}\overset{\leftrightarrow}{T}(\mu_{\mathrm{ATM}(A)}(x),\mu_{\mathrm{ATM}(B)}(x))\big)\\
&= \inf_{x\in X}\min\big(\overset{\leftrightarrow}{T}(\mu_{\mathrm{ATL}(A)}(x),\mu_{\mathrm{ATL}(B)}(x)),\\
&\qquad\qquad\quad \overset{\leftrightarrow}{T}(\mu_{\mathrm{ATM}(A)}(x),\mu_{\mathrm{ATM}(B)}(x))\big)\\
&\leq \inf_{x\in X}\overset{\leftrightarrow}{T}\big(\min(\mu_{\mathrm{ATL}(A)}(x),\mu_{\mathrm{ATM}(A)}(x)),\\
&\qquad\qquad \min(\mu_{\mathrm{ATM}(A)}(x),\mu_{\mathrm{ATM}(B)}(x))\big)\\
&= \inf_{x\in X}\overset{\leftrightarrow}{T}\big(\mu_{\mathrm{ECX}(A)}(x),\mu_{\mathrm{ECX}(B)}(x)\big)\\
&= \mathrm{SIM}_T\big(\mathrm{ECX}(A),\mathrm{ECX}(B)\big)
\end{aligned}$$

□

One would naturally assume equality to hold in (6.6) in direct analogy to the crisp case (see Theorem 6.3)

$$A \lesssim_L B \wedge A \gtrsim_L B \iff \mathrm{ECX}(A)=\mathrm{ECX}(B).$$

The choice of the connecting t-norm $\tilde{T}$, however, is the crucial point in this question. As apparent from the above proof, $T_{\mathbf{M}}$ could be the right choice. Before going into more detail, we have to prove a fundamental prerequisite which states that inclusion and similarity of fuzzy sets are preserved by any hull or image operator—including also the extension principle [26].

6.10. Lemma. *Hulls/images with respect to arbitrary fuzzy relations are "inclusion-preserving", i.e. for all fuzzy subsets $A,B\in\mathcal{F}(X)$ and all binary fuzzy relations R on X, the following holds:*

$$\mathrm{INCL}_T(A,B)\leq \mathrm{INCL}_T(H_R(A),H_R(B))$$

Moreover, they are "similarity-preserving", that is, with the above assumptions,

$$\mathrm{SIM}_T(A,B)\leq \mathrm{SIM}_T(H_R(A),H_R(B)).$$

Proof. According to Point 8. of Lemma 2.32, we can expand the definition with $R(y,x)$:

$$\begin{aligned}
\mathrm{INCL}_T(A,B) &= \inf_{y\in X}\vec{T}\big(\mu_A(y),\mu_B(y)\big)\\
&\leq \inf_{y\in X}\vec{T}\big(T(\mu_A(y),R(y,x)),T(\mu_B(y),R(y,x))\big)
\end{aligned}$$

Since the above inequality holds for all $x \in X$, we can even take the infimum and obtain

$$\mathrm{INCL}_T(A,B) \leq \inf_{x\in X} \inf_{y\in X} \vec{T}\big(T(\mu_A(y), R(y,x)), T(\mu_B(y), R(y,x))\big).$$

Since $\vec{T}$ is non-decreasing in its second argument, the first assertion follows just by taking the supremum in the second argument and taking into account that $\vec{T}$ is left-continuous in its first argument (cf. Lemma 2.33):

$$\begin{aligned}
&\inf_{x\in X} \inf_{y\in X} \vec{T}\big(T(\mu_A(y), R(y,x)), T(\mu_B(y), R(y,x))\big) \\
&\qquad \leq \inf_{x\in X} \inf_{y\in X} \vec{T}\big(T(\mu_A(y), R(y,x)), \sup_{z\in X} T(\mu_B(z), R(z,x))\big) \\
&\qquad \leq \inf_{x\in X} \vec{T}\big(\sup_{y\in X} T(\mu_A(y), R(y,x)), \sup_{z\in X} T(\mu_B(z), R(z,x))\big) \\
&\qquad = \inf_{x\in X} \vec{T}\big(\mu_{H_R(A)}(x), \mu_{H_R(B)}(x)\big) \\
&\qquad = \mathrm{INCL}_T(H_R(A), H_R(B))
\end{aligned}$$

The same is, of course, obtained if A and B are swapped and we get

$$\begin{aligned}
\mathrm{SIM}_T(A,B) &= \min\big(\mathrm{INCL}_T(A,B), \mathrm{INCL}(B,A)\big) \\
&\leq \min\big(\mathrm{INCL}_T(H_R(A), H_R(B)), \mathrm{INCL}(H_R(B), H_R(A))\big) \\
&= \mathrm{SIM}_T\big(H_R(A), H_R(B)\big)
\end{aligned}$$

which completes the proof. □

Now we are able to show that, for $\tilde{T} = T_{\mathbf{M}}$, we have a direct analogue of Theorem 6.3 in the sense that equality holds in (6.6).

6.11. Corollary. *The fuzzy relation*

$$\begin{aligned}
\mathcal{L}_L(A,B) = \mathcal{L}_{T_{\mathbf{M}},L}(A,B) = \min\big(&\mathrm{INCL}_T(\mathrm{ATL}(B), \mathrm{ATL}(A)), \\
&\mathrm{INCL}_T(\mathrm{ATM}(A), \mathrm{ATM}(B))\big)
\end{aligned}$$

is a T-$\mathcal{E}_L$-ordering, where

$$\mathcal{E}_L(A,B) = \mathrm{SIM}_T\big(\mathrm{ECX}(A), \mathrm{ECX}(B)\big).$$

Proof. We already know from (6.6) that

$$\mathcal{E}_{T_{\mathbf{M}},L}(A,B) \leq \mathrm{SIM}_T\big(\mathrm{ECX}(B), \mathrm{ECX}(A)\big).$$

Lemma 6.10 provides the basis for applying a similar argument as in the proof of Theorem 6.3 in the fuzzy case:

$$\begin{aligned}\mathrm{SIM}_T\big(\mathrm{ECX}(A),\mathrm{ECX}(B)\big) &\leq \mathrm{SIM}_T\big(\mathrm{ATL}(\mathrm{ECX}(A)),\mathrm{ATL}(\mathrm{ECX}(B))\big)\\ &= \mathrm{SIM}_T\big(\mathrm{ATL}(A),\mathrm{ATL}(B))\big)\\ \mathrm{SIM}_T\big(\mathrm{ECX}(A),\mathrm{ECX}(B)\big) &\leq \mathrm{SIM}_T\big(\mathrm{ATM}(\mathrm{ECX}(A)),\mathrm{ATM}(\mathrm{ECX}(B))\big)\\ &= \mathrm{SIM}_T\big(\mathrm{ATM}(A),\mathrm{ATM}(B))\big)\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{E}_{T_{\mathbf{M}},L}(A,B) &= \min\big(\mathrm{SIM}_T(\mathrm{ATL}(A),\mathrm{ATL}(B)),\mathrm{SIM}_T(\mathrm{ATM}(A),\mathrm{ATM}(B))\big)\\ &\geq \mathrm{SIM}_T\big(\mathrm{ECX}(B),\mathrm{ECX}(A)\big)\end{aligned}$$

and the proof is completed. □

It remains an open question in which way the crisp ordering $\lesssim_L$ and the fuzzy ordering $\mathcal{L}_{\tilde{T},L}$ are related to each other, in particular, whether $\mathcal{L}_{\tilde{T},L}$ is a fuzzification of $\lesssim_L$ in the sense of Definition 3.4. The next result gives an exhaustive answer.

6.12. Proposition. *For an arbitrary t-norm $\tilde{T}$ which dominates T, the following characterization of the kernel of $\mathcal{L}_{\tilde{T},L}$ is obtained:*

$$\forall A,B\in\mathcal{F}(X):\ \mathcal{L}_{\tilde{T},L}(A,B)=1 \iff A\lesssim_L B \tag{6.7}$$

Moreover, $\mathcal{E}_{\tilde{T},L}$ is separated on $\mathcal{F}_L(X)$, i.e.

$$\forall A,B\in\mathcal{F}(X):\ \mathcal{E}_{\tilde{T},L}(A,B)=1 \iff A\cong_L B \tag{6.8}$$

Proof. One easily verifies, taking basic properties of $\vec{T}$ into account, that the following holds for arbitrary $A,B\in\mathcal{F}(X)$:

$$\mathrm{INCL}_T(A,B) \iff A\subseteq B$$

Both assertions follow, then, immediately:

$$\begin{aligned}
\mathcal{L}_{\tilde{T},L}(A,B)=1 &\iff \tilde{T}\big(\mathrm{INCL}_T(\mathrm{ATL}(B),\mathrm{ATL}(A)),\\
&\qquad \mathrm{INCL}_T(\mathrm{ATM}(A),\mathrm{ATM}(B))\big)=1\\
&\iff \mathrm{INCL}_T(\mathrm{ATL}(B),\mathrm{ATL}(A))=1\wedge\\
&\qquad \mathrm{INCL}_T(\mathrm{ATM}(A),\mathrm{ATM}(B))=1\\
&\iff \mathrm{ATL}(B)\subseteq\mathrm{ATL}(A)\wedge\mathrm{ATM}(A)\subseteq\mathrm{ATM}(B)\\
&\iff A\lesssim_L B\\
\mathcal{E}_{\tilde{T},L}(A,B)=1 &\iff \tilde{T}\big(\mathrm{SIM}_T(\mathrm{ATL}(B),\mathrm{ATL}(A)),\\
&\qquad \mathrm{SIM}_T(\mathrm{ATM}(A),\mathrm{ATM}(B))\big)=1\\
&\iff \mathrm{SIM}_T(\mathrm{ATL}(B),\mathrm{ATL}(A))=1\wedge\\
&\qquad \mathrm{SIM}_T(\mathrm{ATM}(A),\mathrm{ATM}(B))=1\\
&\iff \mathrm{ATL}(A)=\mathrm{ATL}(B)\wedge\mathrm{ATM}(A)=\mathrm{ATM}(B)\\
&\iff \mathrm{ECX}(A)=\mathrm{ECX}(B)\\
&\iff A\cong_L B
\end{aligned}$$

□

The equivalence (6.7) states that $\lesssim_L$ is exactly the kernel of $\mathcal{L}_{\tilde{T},L}$. Therefore, we can deduce with Proposition 4.22 that $\mathcal{L}_{\tilde{T},L}$ fuzzifies $\lesssim_L$. Furthermore, the validity of (6.8) means that $\cong_L$ coincides with the kernel of $\mathcal{E}_{\tilde{T},L}$, which implies, due to Proposition 3.7, that $\mathcal{E}_{\tilde{T},L}$ fuzzifies $\cong_L$.

Almost needless to say, all above constructions can also be carried out for $\lesssim_I$ with obvious modifications.

6.13. Example. Let us reconsider the example shown in Figure 6.1, where we denote the left fuzzy quantity with A and the right one with B, i.e.

$$\begin{aligned}
\mu_A(x)&=\max\big(0,1-2\cdot|x-1|\big),\\
\mu_B(x)&=\max\big(0,1-\tfrac{1}{2}\cdot|x-2.3|\big).
\end{aligned}$$

For $L=\chi_\leq$ (which entails $\lesssim_L\equiv\lesssim_I$, etc.), $T=T_{\mathbf{L}}$, and $\tilde{T}=T_{\mathbf{M}}$, we obtain

$$\begin{aligned}
\mathcal{L}_L(A,B)&=0.9\\
\mathcal{L}_L(B,A)&=0\\
\mathcal{E}_L(A,B)&=0
\end{aligned}$$

which seems to be quite a reasonable result.

6.3.2 Compensating different Heights

We have seen already that equal heights of two fuzzy sets are necessary for comparability with respect to $\lesssim_L$, where the same is true for $\lesssim_I$. This is not just an eyesore but really a strong limitation in terms of applicability as pointed out by B. Moser and L. T. Kóczy in several very fruitful and constructive discussions. In accordance to the results in 6.2.2, both suggested to consider α-cuts, but only up to the minimum of the two heights. Undoubtedly, returning to α-cuts would mean a step back, because they are much more complicated to handle and almost intractable when it concerns real implementations. Moreover, a fuzzification as in the previous subsection would be nearly impossible.

In the following, we will present a rather simple way to follow the above suggestions substantially without running into the difficulties which arise when considering α-cuts.

6.14. Definition. The *greatest common level* of two fuzzy subsets A, B of some domain X is defined as

$$\mathrm{gcl}(A, B) = \min\big(\mathrm{height}(A), \mathrm{height}(B)\big).$$

For a fuzzy set A and an $\alpha \in [0, 1]$, the *α-base* $\lfloor A \rfloor_\alpha$ is represented by the membership function

$$\mu_{\lfloor A \rfloor_\alpha}(x) = \min\big(\mu_A(x), \alpha\big).$$

Basically, the only thing, which has to be done in order to allow comparisons of fuzzy sets A, B with different heights, is to replace A and B by their $\mathrm{gcl}(A, B)$-bases. Obviously, this is equivalent to considering inclusions on the level of α-cuts up to the greatest common value, as suggested by Moser and Kóczy. Hence, we can define the following relation on $\mathcal{F}(X)$:

$$A \lesssim'_L B \iff \lfloor A \rfloor_{\mathrm{gcl}(A,B)} \lesssim_L \lfloor B \rfloor_{\mathrm{gcl}(A,B)}$$

It is easy to see that $\lesssim'_L$ is reflexive. Moreover, non-antisymmetry is characterized as

$$A \lesssim'_L B \wedge B \lesssim'_L A \iff \mathrm{ECX}\big(\lfloor A \rfloor_{\mathrm{gcl}(A,B)}\big) = \mathrm{ECX}\big(\lfloor B \rfloor_{\mathrm{gcl}(A,B)}\big).$$

The example shown in Figure 6.3 shows that $\lesssim'_L$ is, however, not transitive. The following holds for these three fuzzy quantities A, B, and C under the assumptions $L = \chi_{\leq}$, $T = T_{\mathbf{L}}$, and $\tilde{T} = T_{\mathbf{M}}$:

$$A \lesssim'_L B$$
$$B \lesssim'_L C$$
$$A \not\lesssim'_L C$$

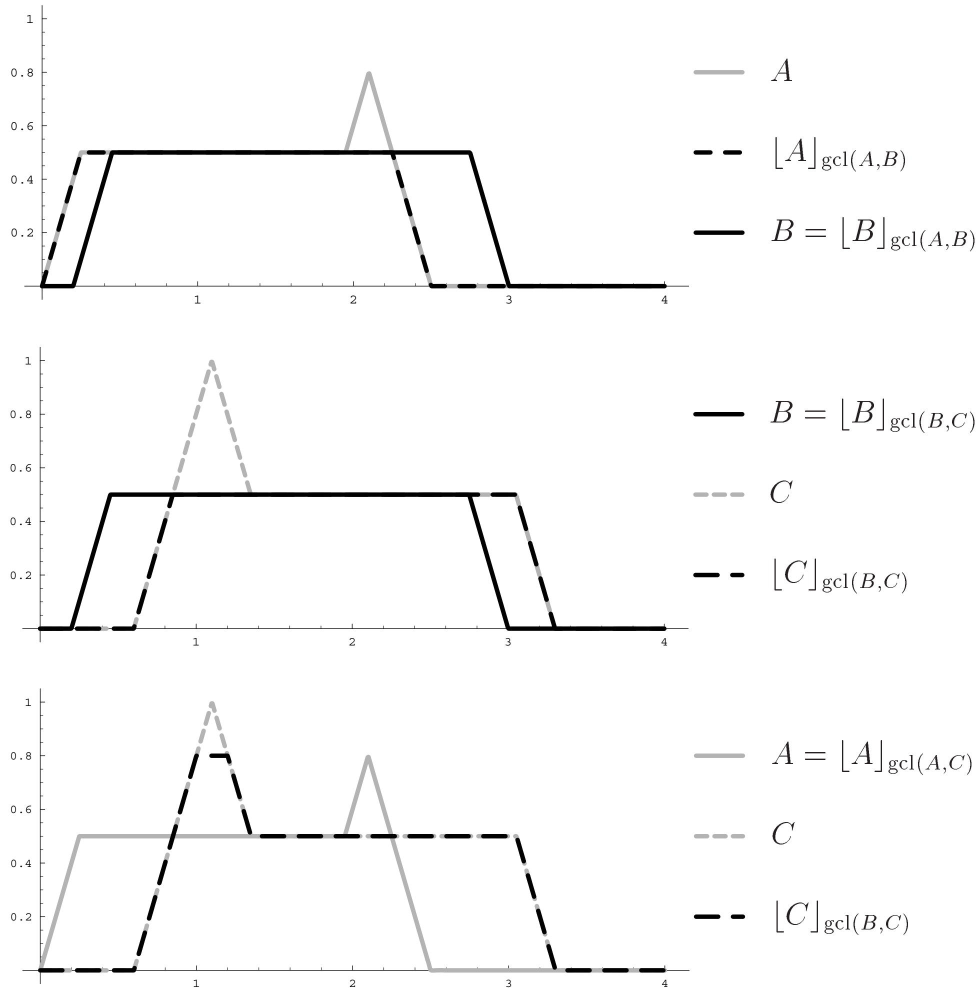


Figure 6.3: A counterexample that transitivity does not hold for $\precsim'_L$.

If we consider the so-called strict part of $\lesssim'_L$, defined as

$$A \lnsim'_L B \iff A \lesssim'_L B \wedge B \not\lesssim'_L A,$$

we see that not even this relation is transitive, since $A \lnsim'_L B$ and $B \lnsim'_L C$.

Apparently, the problem is due to the "interaction" between $\lfloor A \rfloor_{\mathrm{gcl}(A,B)}$ and $\lfloor B \rfloor_{\mathrm{gcl}(A,B)}$, where both sets depend on the shape, more specifically, the height of the respective other one.

The problem could be solved in the framework of our approach if we find a way to force two fuzzy sets to be comparable by compensating the different heights independently—without interacting with any other fuzzy sets. The simplest approach would be to divide each membership value by the height which, obviously, yields a fuzzy set with a height of 1. These divisions, however, cause distortions. Since, from the intuitive point of view, the position of the ceiling plays a fundamental role when ordering fuzzy sets, it would be better just to lift the ceiling such that a fuzzy set with height 1 is obtained.

6.15. Definition. For all $A \in \mathcal{F}(X)$, we define the fuzzy set $\lceil A \rceil$ by the membership function

$$\mu_{\lceil A \rceil}(x) = \begin{cases} 1 & \text{if } \mu_A(x) = \mathrm{height}(A), \\ \mu_A(x) & \text{otherwise.} \end{cases}$$

The next lemma shows in which way this "lifting" of the ceiling guarantees equal heights.

6.16. Lemma. *Provided that $A \in \mathcal{F}_T(X) \cup \mathcal{F}_H(X)$, the following holds:*

$$\mathrm{height}(\lceil A \rceil) = 1$$

Proof. If $A \in \mathcal{F}_H(X)$, the equality $A = \lceil A \rceil$ holds trivially. If, on the other hand, $\mathrm{ceil}(A) \neq \emptyset$ then $\lceil A \rceil$ is normal. □

Now we can define a modified relation $\lesssim''_L$ on $\mathcal{F}_T(X) \cup \mathcal{F}_H(X)$. Due to the previous lemma, incomparabilities, which are caused by different heights, can no longer occur:

$$A \lesssim''_L B \iff \lceil A \rceil \lesssim_L \lceil B \rceil$$

6.17. Proposition. *The relation $\lesssim''_L$ is a preordering on $\mathcal{F}_T(X) \cup \mathcal{F}_H(X)$. Its symmetric kernel $\cong''_L$ is represented as*

$$A \cong''_L B \iff \mathrm{ECX}\big(\lceil A \rceil\big) = \mathrm{ECX}\big(\lceil B \rceil\big).$$

Proof. Follows directly by applying Theorem 6.3 to $\lceil A \rceil$ and $\lceil B \rceil$. □

Obviously, the relations $\lesssim_L$ and $\lesssim''_L$ are equivalent to each other as long as fuzzy sets with a height of 1 are considered. Note that this is not necessarily true for two fuzzy sets with equal heights which are below 1 (that, however, would have been the case for $\lesssim'_L$).

Fuzzification can be carried out analogously, too. All the results, which we have proved so far, still hold with obvious modifications.

6.18. Proposition. *Let $\tilde{T}$ be a t-norm which dominates T. Then the binary fuzzy relation*

$$\begin{aligned} \mathcal{L}''_{\tilde{T},L}(A,B) = \tilde{T}\big(&\mathrm{INCL}_T(\mathrm{ATL}(\lceil B \rceil), \mathrm{ATL}(\lceil A \rceil)),\\ &\mathrm{INCL}_T(\mathrm{ATM}(\lceil A \rceil), \mathrm{ATM}(\lceil B \rceil))\big) \end{aligned}$$

is a T-$\mathcal{E}''_{\tilde{T},L}$ ordering on $\mathcal{F}_T(X) \cup \mathcal{F}_H(X)$ with

$$\begin{aligned} \mathcal{E}''_{\tilde{T},L}(A,B) = \tilde{T}\big(&\mathrm{SIM}_T(\mathrm{ATL}(\lceil A \rceil), \mathrm{ATL}(\lceil B \rceil)),\\ &\mathrm{SIM}_T(\mathrm{ATM}(\lceil A \rceil), \mathrm{ATM}(\lceil B \rceil))\big). \end{aligned}$$

Furthermore, the following inequality holds for all $A, B \in \mathcal{F}_T(X) \cup \mathcal{F}_H(X)$:

$$\mathcal{E}''_{\tilde{T},L}(A,B) \leq \mathrm{SIM}_T\big(\mathrm{ECX}(\lceil A \rceil), \mathrm{ECX}(\lceil B \rceil)\big)$$

Proof. Immediate consequence of Theorem 6.9. □

6.19. Proposition. *The binary fuzzy relation defined as*

$$\begin{aligned} \mathcal{L}''_L(A,B) = \min\big(&\mathrm{INCL}_T(\mathrm{ATL}(\lceil B \rceil), \mathrm{ATL}(\lceil A \rceil)),\\ &\mathrm{INCL}_T(\mathrm{ATM}(\lceil A \rceil), \mathrm{ATM}(\lceil B \rceil))\big) \end{aligned}$$

is a T-$\mathcal{E}''_L$-ordering on $\mathcal{F}_T(X) \cup \mathcal{F}_H(X)$, where

$$\mathcal{E}''_L(A,B) = \mathrm{SIM}_T\big(\mathrm{ECX}(\lceil A \rceil), \mathrm{ECX}(\lceil B \rceil)\big).$$

Proof. Immediate consequence of Corollary 6.11. □

6.3.3 A Clue to Hybridization

If we assume that there are applications in which it is not appropriate to consider only (extensional) convex fuzzy sets, the above approaches—both crisp and fuzzy—could run into problems, simply because of their inability to distinguish between fuzzy alternatives with equal (extensional) convex hulls. As already mentioned, the two fuzzy quantities in Figure 6.2 demonstrate, that the (non-)antisymmetry of the preordering $\lesssim_L$ is not fully exhaustive.

One possible way to gain "more" antisymmetry while keeping all our present achievements is hybridization with some other ordering method. More specifically, we have already seen in 4.4.1 (compare with the proof of Theorem 4.26) that it is possible to construct an ordering $\lesssim$ from a preordering $\trianglelefteq$ such that $\lesssim$ is a maximal subrelation of $\trianglelefteq$. This is achieved by factorization with respect to the symmetric kernel of $\trianglelefteq$, defining orderings of each equivalence class[1], and lexicographic composition.

Such a construction can be applied to a preordering $\trianglelefteq$ even if preorderings of the equivalence classes with respect to its symmetric kernel are considered, with the result that the lexicographic composition does not necessarily yield antisymmetry. Nevertheless, this can still be an improvement if the preorderings of the equivalence classes are more specific than $\trianglelefteq$.

In order to see what is obtained if we apply this construction principle to $\lesssim_L$ (can be transferred to $\lesssim_I$ and $\lesssim_L''$ with obvious modifications), let us consider another preordering of fuzzy sets $\preceq$. If we denote the greatest subclass of fuzzy subsets, such that both methods can be applied, with $\mathcal{S}$, the relation

$$A \precsim_L B \iff \big((A \cong_L B \wedge A \preceq B) \vee (A \lesssim_L B \wedge B \not\lesssim_L A)\big)$$

is a preordering on $\mathcal{S}$, where the following properties hold obviously:

1. If $\preceq$ is an ordering, $\precsim_L$ is an ordering.
2. The lexicographic composition $\precsim_L$ is a subrelation of $\lesssim_L$, i.e. more specific than $\lesssim_L$:

   $$\forall A, B \in \mathcal{S}: \ A \precsim_L B \Longrightarrow A \lesssim_L B$$

   This statement also entails that the original preordering $\lesssim_L$ has priority over $\preceq$.

[1]If there is no hint how to choose the orderings of the equivalence classes, the well-ordering theorem still guarantees the existence of such orderings. This is, however, due to the axiom of choice and, therefore, no constructive answer.

3. The symmetric kernel is the intersection of the symmetric kernels of the two relations, i.e. for all $A, B \in \mathcal{S}$,

$$A \precsim_L B \wedge B \precsim_L A \iff A \cong_L B \wedge A \preceq B \wedge B \preceq A,$$

which means that $\precsim_L$ is "at least as antisymmetric" as $\lesssim_L$ and $\preceq$.

There is, however, one important aspect that should not be overlooked. If $\preceq$ is not linear, we can come to the peculiar situation that two fuzzy sets are treated as equal by $\lesssim_L$ but incomparable with respect to $\precsim_L$.

In the case of the real numbers $X = \mathbb{R}$, a possibility could be to use one of the known ranking methods which is reflexive and transitive. The simplest case could be a method which orders fuzzy quantities according to one characteristic value (methods of the first class in terms of Wang's and Kerre's classification [66, 68]). Of course, mapping a fuzzy set to one single value results in a dramatic loss of information as already pointed out by Freeling [21], Wang, and Kerre [68]. However, the influence of this loss is, in this specific case, limited, because we are only considering fuzzy quantities with equal (extensional) convex hulls. The only crucial thing is whether the method can yield an improvement at all. Adamo's method of considering the rightmost value of some α-cut [1], for instance, is even less specific than the relation $\lesssim_I$. The same happens with Fortemps' and Roubens' approach, which also considers just the boundaries of α-cuts [19], and other methods which are only applicable for convex fuzzy quantities [10, 43]. Thinking of the example in Figure 6.2, comparing the centers of gravity of the membership functions [8] would do a perfect job, the Yager indices [71] as well.

Beside the above approaches, there are a lot of other methods for ordering fuzzy quantities, some computing distances to reference sets [9, 30, 31], some utilizing fuzzy relations [3, 13, 54]. It is beyond the scope of this thesis to check the properties of all possible combinations—the demands on such a method depend on the specific application anyway.

Finally, it is worth to mention that there is neither a theoretical nor a practical obstacle to repeat hybridization once or even more often. This iterative process always yields a preordering as long as all relations, which are composed, are preorderings.

6.4 The Monotonicity of Extended Monotonic Mappings

Now we want to clarify in which way the extensions of monotonic mappings are monotonic with respect to the two basic preorderings of fuzzy sets we have discussed so far. For this purpose, we consider n-ary functions of the type

$$\varphi : \quad X_1 \times \cdots \times X_n \quad \longrightarrow \quad Y,$$

with the following additional assumptions:

1. Each X_i is equipped with a strongly linear T-E_i-ordering L_i which directly fuzzifies a crisp linear ordering $\lesssim_i$. By means of the construction (6.3), $\lesssim_i$ induces a preordering on $\mathcal{F}(X_i)$ which we denote with $\lesssim_{I_i}$. According to (6.4), L_i induces a preordering on $\mathcal{F}(X_i)$ which will be called $\lesssim_{L_i}$.

2. The range space Y is equipped with a strongly linear T-E_y-ordering L_y which directly fuzzifies the crisp linear ordering $\lesssim_y$. As above, let us denote the two preorderings of fuzzy sets with $\lesssim_{I_y}$ and $\lesssim_{L_y}$, respectively.

3. All partial mappings are non-decreasing, i.e. for all $i = 1, \ldots, n$ and all $x_i, x_i' \in X_i$, the implication

$$x_i \lesssim_i x_i' \Longrightarrow \varphi(x_1, \ldots, x_i, \ldots, x_n) \lesssim_y \varphi(x_1, \ldots, x_i', \ldots, x_n)$$

holds, where the values $x_j \in X_j$, for $j \neq i$, are arbitrary but fixed.

The question is whether the partial mappings of the extension of φ are also non-decreasing with respect to the above preorderings of fuzzy sets. Before going into detail, let us briefly mention an auxiliary representation of inclusions between continuations which will be helpful in our further investigations.

6.20. Lemma. *The following holds for arbitrary fuzzy subsets A, B of an ordered domain $(X, \lesssim)$:*

$$\mathrm{LTR}(A) \subseteq \mathrm{LTR}(B) \iff \forall \alpha \in [0,1) \; \forall x \in [A]_{\underline{\alpha}} \; \exists y \in [B]_{\underline{\alpha}} : y \lesssim x$$
$$\mathrm{RTL}(A) \subseteq \mathrm{RTL}(B) \iff \forall \alpha \in [0,1) \; \forall x \in [A]_{\underline{\alpha}} \; \exists y \in [B]_{\underline{\alpha}} : x \lesssim y$$

Proof. From Lemma 2.6 and Theorem 5.12, we obtain

$$\begin{aligned} \mathrm{LTR}(A) \subseteq \mathrm{LTR}(B) &\iff \forall \alpha \in [0,1) : \ [\mathrm{LTR}(A)]_{\underline{\alpha}} \subseteq [\mathrm{LTR}(B)]_{\underline{\alpha}} \\ &\iff \forall \alpha \in [0,1) : \ \mathrm{LTR}\big([A]_{\underline{\alpha}}\big) \subseteq \mathrm{LTR}\big([B]_{\underline{\alpha}}\big) \end{aligned}$$

and the assertion for LTR follows trivially with Lemma 6.1. The same argument can be applied analogously to prove the corresponding equivalence for RTL. □

The following result already gives a positive answer that monotonicities are preserved by $T_{\mathbf{M}}$-extensions if considering preorderings of type (6.3).

6.21. Theorem. *All partial mappings of the $T_{\mathbf{M}}$-extension of φ, denoted as $\hat{\varphi}$, are non-decreasing, i.e. for all $i = 1, \dots, n$ and all $A_i, A_i' \in \mathcal{F}(X_i)$, the implication*

$$A_i \lesssim_{I_i} A_i' \Longrightarrow \hat{\varphi}(A_1, \dots, A_i, \dots, A_n) \lesssim_{I_y} \hat{\varphi}(A_1, \dots, A_i', \dots, A_n)$$

holds, where, for $j \neq i$, the other components $A_j \in \mathcal{F}(X_j)$ are arbitrary but fixed.

Proof. Let us assume that $A_i \lesssim_{I_i} A_i'$ holds for some i and some fuzzy subsets $A_i, A_i' \in \mathcal{F}(X_i)$. This implies

$$\mathrm{LTR}(A_i') \subseteq \mathrm{LTR}(A_i)$$

and further, due to Lemma 6.20,

$$\forall \alpha \in [0,1) \ \forall x_i' \in [A_i']_{\underline{\alpha}} \ \exists x_i \in [A_i]_{\underline{\alpha}} : \ x_i \lesssim x_i'. \tag{6.9}$$

Now take an arbitrary $\alpha \in [0,1)$ and a value $y' \in [\hat{\varphi}(A_1, \dots, A_i', \dots, A_n)]_{\underline{\alpha}}$, i.e. according to Definition 2.41,

$$\begin{aligned} \sup \big\{ \min \big(\mu_{A_i}(x_1), \dots, \mu_{A_i'}(x_i'), \dots, \mu_{A_n}(x_n)\big) \mid \\ \varphi(x_1, \dots, x_i', \dots, x_n) = y' \big\} > \alpha. \end{aligned}$$

Therefore, we can choose a vector $(x_1, \dots, x_i', \dots, x_n)$ such that

$$\varphi(x_1, \dots, x_i', \dots, x_n) = y'$$

and

$$\min \big(\mu_{A_i}(x_1), \dots, \mu_{A_i'}(x_i'), \dots, \mu_{A_n}(x_n)\big) > \alpha.$$

In particular, $\mu_{A'_i}(x'_i) > \alpha$ must hold. From (6.9), we know that there exists an $x_i \in [A_i]_{\underline{\alpha}}$ with $x_i \lesssim_i x'_i$ holds. Of course, the following is true as well:

$$\min\big(\mu_{A_i}(x_1), \ldots, \mu_{A_i}(x_i), \ldots, \mu_{A_n}(x_n)\big) > \alpha$$

Since all partial mappings of φ are non-decreasing, we obtain that

$$y = \varphi(x_1, \ldots, x_i, \ldots, x_n) \lesssim_y \varphi(x_1, \ldots, x'_i, \ldots, x_n) = y'.$$

So, we have found a $y \lesssim_y y'$ in $[\hat{\varphi}(A_1, \ldots, A_i, \ldots, A_n)]_{\underline{\alpha}}$, which implies, with Lemma 6.20,

$$\mathrm{LTR}\big(\hat{\varphi}(A_1, \ldots, A'_i, \ldots, A_n)\big) \subseteq \mathrm{LTR}\big(\hat{\varphi}(A_1, \ldots, A_i, \ldots, A_n)\big).$$

The same argument can be applied analogously to show

$$\mathrm{RTL}\big(\hat{\varphi}(A_1, \ldots, A_i, \ldots, A_n)\big) \subseteq \mathrm{RTL}\big(\hat{\varphi}(A_1, \ldots, A'_i, \ldots, A_n)\big)$$

and the proof is completed. □

A consequence of the previous theorem is that the extensions of isotonic operations are isotonic with respect to $\lesssim_I$. More specifically, if $n = 2$, $X_1 = X_2 = Y$, where $*$ is an isotonic operation on X, i.e.

$$\forall x, y, z \in X: \ \big(y \lesssim z \Rightarrow x * y \lesssim x * z\big) \wedge \big(x \lesssim y \Rightarrow x * z \lesssim y * z\big),$$

the extension of $*$ to fuzzy subsets, denoted $\hat{*}$, is also isotonic with respect to $\lesssim_I$:

$$\begin{aligned}\forall A, B, C \in \mathcal{F}(X): \ &\big(B \lesssim_I C \Rightarrow A \hat{*} B \lesssim_I A \hat{*} C\big) \wedge \\ &\big(A \lesssim_I B \Rightarrow A \hat{*} C \lesssim_I B \hat{*} C\big)\end{aligned}$$

In particular, this means that additions of fuzzy quantities are always isotonic with respect to the preordering of fuzzy quantities induced by the crisp ordering of real numbers. The same is true for the multiplication of positive fuzzy quantities.

The question remains whether the same results hold if we admit indistinguishability. As a first immediate result, the answer is positive under some restrictions in terms of extensionality.

6.22. Corollary. *For all $i = 1, \ldots, n$ and all extensional fuzzy subsets $A_i, A'_i \in \mathcal{F}(X_i)$, the implication*

$$A_i \lesssim_{L_i} A'_i \Longrightarrow \hat{\varphi}(A_1, \ldots, A_i, \ldots, A_n) \lesssim_{L_y} \hat{\varphi}(A_1, \ldots, A'_i, \ldots, A_n)$$

holds, where the other components $A_j \in \mathcal{F}(X_j)$, for $j \neq i$, are arbitrary but fixed (not necessarily extensional).

Proof. For some $i = 1, \ldots, n$, assume that we have two extensional fuzzy sets A_i, A'_i for which $A_i \lesssim_{L_i} A'_i$ holds. Due to Theorem 5.3, we obtain the following:

$$\begin{aligned} A_i \lesssim_{L_i} A'_i &\iff \mathrm{LTR}(\mathrm{EXT}(A_i)) \supseteq \mathrm{LTR}(\mathrm{EXT}(A'_i)) \wedge \\ & \mathrm{RTL}(\mathrm{EXT}(A_i)) \subseteq \mathrm{RTL}(\mathrm{EXT}(A'_i)) \\ &\iff \mathrm{LTR}(A_i) \supseteq \mathrm{LTR}(A'_i) \wedge \mathrm{RTL}(A_i) \subseteq \mathrm{RTL}(A'_i) \end{aligned}$$

Theorem 6.21 implies

$$\begin{aligned} \mathrm{LTR}\big(\hat{\varphi}(A_1, \ldots, A_i, \ldots, A_n)\big) &\supseteq \mathrm{LTR}\big(\hat{\varphi}(A_1, \ldots, A'_i, \ldots, A_n)\big), \\ \mathrm{RTL}\big(\hat{\varphi}(A_1, \ldots, A_i, \ldots, A_n)\big) &\subseteq \mathrm{RTL}\big(\hat{\varphi}(A_1, \ldots, A'_i, \ldots, A_n)\big). \end{aligned}$$

Since, with Corollary 2.55, applying EXT does not change the above inclusions, we obtain

$$\begin{aligned} \mathrm{EXT}\big(\mathrm{LTR}(\hat{\varphi}(A_1, \ldots, A_i, \ldots, A_n))\big) &\supseteq \mathrm{EXT}\big(\mathrm{LTR}(\hat{\varphi}(A_1, \ldots, A'_i, \ldots, A_n))\big), \\ \mathrm{EXT}\big(\mathrm{RTL}(\hat{\varphi}(A_1, \ldots, A_i, \ldots, A_n))\big) &\subseteq \mathrm{EXT}\big(\mathrm{RTL}(\hat{\varphi}(A_1, \ldots, A'_i, \ldots, A_n))\big). \end{aligned}$$

The fuzzy ordering L_y is a direct fuzzification of $\lesssim_y$. Theorem 5.3 and Remark 5.4, therefore, imply

$$\begin{aligned} \mathrm{ATL}\big(\hat{\varphi}(A_1, \ldots, A_i, \ldots, A_n)\big) &\supseteq \mathrm{ATL}\big(\hat{\varphi}(A_1, \ldots, A'_i, \ldots, A_n)\big), \\ \mathrm{ATM}\big(\hat{\varphi}(A_1, \ldots, A_i, \ldots, A_n)\big) &\subseteq \mathrm{ATM}\big(\hat{\varphi}(A_1, \ldots, A'_i, \ldots, A_n)\big), \end{aligned}$$

and we have proved the assertion. $\square$

Already a very simple example shows that the assertion of Corollary 6.22 does not hold any longer if the assumption concerning extensionality is omitted. The reason is that it can happen that significant differences, which are masked by the extensional hull, may be made visible by φ. It may be clear, that the crucial point is whether extensionality is preserved by the mapping φ. The following result gives an ultimate answer for the case $n = 1$.

6.23. Theorem. *Consider a non-decreasing mapping $\varphi : X \to Y$. Then the implication*

$$A \lesssim_{L_x} B \implies \hat{\varphi}(A) \lesssim_{L_y} \hat{\varphi}(B)$$

holds for any two fuzzy subsets $A, B \in \mathcal{F}(X)$ if and only if φ is "similarity-preserving[2]", i.e.

$$\forall x_1, x_2 \in X : \; E_x(x_1, x_2) \leq E_y(\varphi(x_1), \varphi(x_2)). \tag{6.10}$$

[2] Preservation of similarity can also be regarded as a kind of generalized continuity. Another common term in literature is "extensionality" (of mappings) [40].

Proof. Assume that $A \lesssim_{L_x} B$ holds. According to Theorem 5.3 and Lemma 6.20, this implies

$$\forall \alpha \in [0,1)\ \forall x_1 \in [\mathrm{EXT}(B)]_{\underline{\alpha}}\ \exists x_2 \in [\mathrm{EXT}(A)]_{\underline{\alpha}} :\ x_2 \lesssim_x x_1. \tag{6.11}$$

We have to show

$$\forall \alpha \in [0,1)\ \forall y_1 \in [\mathrm{EXT}(\hat{\varphi}(B))]_{\underline{\alpha}}\ \exists y_2 \in [\mathrm{EXT}(\hat{\varphi}(A))]_{\underline{\alpha}} :\ y_2 \lesssim_y y_1.$$

For that purpose, consider an $\alpha \in [0,1)$ and some $y_1 \in [\mathrm{EXT}(\hat{\varphi}(B))]_{\underline{\alpha}}$, i.e.

$$\sup_{y'_1 \in Y} T\big(\mu_{\hat{\varphi}(B)}(y'_1), E_y(y'_1, y_1)\big) > \alpha,$$

which implies

$$\exists y'_1 \in Y :\ T\big(\mu_{\hat{\varphi}(B)}(y'_1), E_y(y'_1, y_1)\big) > \alpha.$$

Let us choose such a y'_1. Of course, $\mu_{\hat{\varphi}(B)}(y'_1) > \alpha$ must hold, from which we obtain

$$\sup\{\mu_B(x_1) \mid \varphi(x_1) = y'_1\} > \alpha \implies \big(\exists x_1 \in X :\ \mu_B(x_1) > \alpha \wedge \varphi(x_1) = y'_1\big).$$

If we choose such an x_1, we obtain, particularly, that $x_1 \in [\mathrm{EXT}(B)]_{\underline{\alpha}}$. Then we know from (6.11) that there must exist an $x_2 \lesssim_x x_1$ contained in $[\mathrm{EXT}(A)]_{\underline{\alpha}}$, which is equivalent to

$$\sup_{x'_2 \in X} T\big(\mu_A(x'_2), E_x(x'_2, x_2)\big) > \alpha.$$

Hence, we can choose such an $x'_2 \in X$ for which

$$T\big(\mu_A(x'_2), E_x(x'_2, x_2)\big) > \alpha$$

holds. Now we define $y_2 = \varphi(x_2)$, where the monotonicity of φ assures that $y_2 \lesssim_y y_1$. Let us consider the membership degree

$$\mu_{\mathrm{EXT}(\hat{\varphi}(A))}(y_2) = \sup_{y'_2 \in Y} T\big(\mu_{\hat{\varphi}(A)}(y'_2), E_y(y'_2, y_2)\big)$$

which is, with $y''_2 = \varphi(x'_2)$, at least as high as

$$T\big(\mu_{\hat{\varphi}(A)}(y''_2), E_y(y''_2, y_2)\big). \tag{6.12}$$

Since $\mu_{\hat{\varphi}(A)}(y''_2) \geq \mu_A(x'_2)$ and, due to (6.10),

$$E_x(x'_2, x_2) \leq E_y(\varphi(x'_2), \varphi(x_2)) = E_y(y''_2, y_2),$$

we obtain that (6.12) is greater or equal to

$$T\big(\mu_A(x_2'), E_x(x_2', x_2)\big) > \alpha,$$

and we have shown that

$$\mathrm{LTR}(A) \supseteq \mathrm{LTR}(B) \implies \mathrm{LTR}(\hat{\varphi}(A)) \supseteq \mathrm{LTR}(\hat{\varphi}(B)).$$

An analogous argument can be used to prove

$$\mathrm{RTL}(A) \subseteq \mathrm{RTL}(B) \implies \mathrm{RTL}(\hat{\varphi}(A)) \subseteq \mathrm{RTL}(\hat{\varphi}(B)),$$

and we finally succeeded in proving that (6.10) is a sufficient condition for the monotonicity of $\hat{\varphi}$.

Reversely, assume that (6.10) does not hold, which means that there exist two values $x_1, x_2 \in X$ such that

$$E_x(x_1, x_2) > E_y(\varphi(x_1), \varphi(x_2)),$$

which also implies $\varphi(x_1) \neq \varphi(x_2)$. Without loss of generality, assume that $x_2 \lesssim_x x_1$. We define the two fuzzy sets $A = \{x_1\}$ and

$$\mu_B(x) = \begin{cases} 1 & \text{if } x = x_1, \\ E(x_1, x_2) & \text{if } x = x_2, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that

$$\mu_{\mathrm{EXT}(A)}(x) = \mu_{\mathrm{EXT}(B)}(x) = E(x_1, x)$$

which implies that $A \lesssim_{L_x} B$. Moreover, one easily verifies the following equalities:

$$\begin{aligned}
\mu_{\hat{\varphi}(A)}(y) &= \chi_{\{\varphi(x_1)\}}(y) \\
\mu_{\hat{\varphi}(B)}(y) &= \begin{cases} 1 & \text{if } y = \varphi(x_1), \\ E(x_1, x_2) & \text{if } y = \varphi(x_2), \\ 0 & \text{otherwise.} \end{cases} \\
\mu_{\mathrm{LTR}(\hat{\varphi}(A))}(y) &= \begin{cases} 1 & \text{if } y \gtrsim_y \varphi(x_1), \\ 0 & \text{otherwise,} \end{cases} \\
\mu_{\mathrm{EXT}(\hat{\varphi}(A))}(y) &= E_y(\varphi(x_1), y).
\end{aligned}$$

With Lemma 5.2, we obtain

$$\begin{aligned}\mu_{\mathrm{ATL}(\hat{\varphi}(A))}(\varphi(x_2)) &= \max\left(\mu_{\mathrm{LTR}(\hat{\varphi}(A))}(\varphi(x_2)), \mu_{\mathrm{EXT}(\hat{\varphi}(A))}(\varphi(x_2))\right)\\ &= \max\left(0, E_y(\varphi(x_1), \varphi(x_2))\right)\\ &= E_y(\varphi(x_1), \varphi(x_2))\\ &< E_x(x_1, x_2) = \mu_{\hat{\varphi}(B)}(\varphi(x_2))\end{aligned}$$

and we see that $\mathrm{ATL}(\hat{\varphi}(A)) \supseteq \mathrm{ATL}(\hat{\varphi}(B))$ cannot be fulfilled, which completes the proof. □

Finally, it is worth to mention that an n-dimensional analogue of Theorem 6.23 does not hold. Consider, for instance, the following example (compare also with Example 4.28):

$$\begin{aligned}X &= \mathbb{R}\\ T &= T_{\mathbf{L}}\\ E(x, y) &= \max(1 - |x - y|, 0)\\ L(x, y) &= \begin{cases} 1 & \text{if } x \le y\\ \max(1 - y + x, 0) & \text{otherwise}\end{cases}\\ \mu_A(x) &= \chi_{\{1\}}(x)\\ \mu_B(x) &= \mu_C(x) = \max(1 - |x - 1|, 0)\\ \varphi(x, y) &= x + y\end{aligned}$$

It is easy to prove that both partial mappings fulfill condition (6.10):

$$\begin{aligned}E(x, y) &= \max(1 - |x - y|, 0)\\ &= \max(1 - |x + z - y - z|, 0) = E(x + z, y + z)\\ E(y, z) &= \max(1 - |y - z|, 0)\\ &= \max(1 - |x + y - x - z|, 0) = E(x + y, x + z)\end{aligned}$$

Let us denote the extended addition of fuzzy quantities with the symbol $\oplus$. Figure 6.4 shows the shape of A, B, C, $A \oplus B$, and $A \oplus C$. Since the equalities $\mathrm{EXT}(A) = \mathrm{EXT}(B) = B$ are valid, the inclusion $A \lesssim_L B$ holds trivially, but $A \oplus B$ and $A \oplus C$ are incomparable (for details how to compute the sum of two fuzzy quantities efficiently we refer to [14, 51, 52]). Moreover, this example shows (C is extensional!) that not even extensionality of the fuzzy sets in the fixed components can guarantee monotonicity.

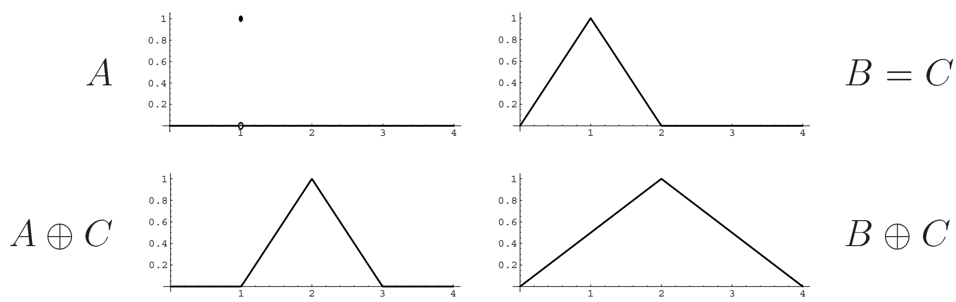


Figure 6.4: A counterexample that partial non-decreasingness does not necessarily imply non-decreasingness of the extension when considering $\lesssim_L$.

6.5 Classification according to Wang and Kerre

Basically because orderings and rankings of fuzzy sets play such an important role in a wide variety of different disciplines, much attention has been paid to these questions in the fuzzy community—resulting in more than 40 different approaches, ranging from extremely simple via fairly complicated to almost intractable ones. So far, none of these methods is commonly accepted, perhaps due to the fact that the debate about their behavior or quality has always been kept on a rather subjective level. In [68], X. Wang and E. E. Kerre argue that

> *" . . . the intuition criterion is extensively applied by researchers. If one tries to develop a new method aiming at the improvement of an established ordering procedure, one normally designs some examples in which the newly developed method derives more reasonable resulting rankings than the known one by his intuition."*

With the motivation to add some objectivity to these discussions, Wang and Kerre [66, 68, 69] proposed a set of intuitively reasonable criteria for judging the properties of various ranking methods for fuzzy quantities. Although there is no one-to-one correspondence between ranking and ordering methods, e.g. concerning linearity, it could still be interesting to check which of the criteria are met by the preorderings $\lesssim_I$ and $\lesssim_L$. So, in the following, let us assume that $\lesssim_I$ is the preordering induced by some crisp ordering $\lesssim$ on X and that $\lesssim_L$ is the preordering corresponding to some fuzzy ordering L which directly fuzzifies $\lesssim$.

Wang's and Kerre's original criteria [66, 68] were formulated for ranking methods. If we rephrase them such that they can be applied meaningfully to a pairwise comparison $\precsim$ on a subclass $\mathcal{S} \subseteq \mathcal{F}(X)$, where X is an arbitrary crisp set, the following is obtained for the first three of them:

$\mathbf{A_1}$ The relation $\precsim$ is reflexive.

$\mathbf{A_2}$ The symmetric kernel of $\precsim$ is an equivalence relation.

$\mathbf{A_3}$ The relation $\precsim$ is transitive.

We know from Theorem 6.3 and Corollary 6.4 that all these three conditions are satisfied for both $\lesssim_L$ and $\lesssim_I$ on $\mathcal{S} = \mathcal{F}(X)$. Furthermore, we have shown in Proposition 6.5 that the two relations are even orderings if we restrict $\mathcal{S}$ to $\mathcal{F}_L(X)$ or $\mathcal{F}_I(X)$, respectively.

The next two criteria state that perfect separation of the supports of two fuzzy sets should imply an order relation between the two fuzzy sets. Originally, these criteria employed infima and suprema of the supports. We will use equivalent formulations, which can even be used in the case of partial orderings or when infima and suprema are not guaranteed to exist:

$\mathbf{A_4}$ The inequality $A \precsim B$ holds for any two fuzzy sets $A, B \in \mathcal{S}$ which have the following property:

$$\exists u, v \in X\ \forall x \in \operatorname{supp}(A)\ \forall y \in \operatorname{supp}(B) :\ x \lnsim u \lnsim v \lnsim y \tag{6.13}$$

$\mathbf{A'_4}$ The strict inequality $A \precnsim B$ holds for any two fuzzy sets $A, B \in \mathcal{S}$ satisfying (6.13).

We have seen in 6.2.3 (cf. Lemma 6.8) that comparability is crucial without restrictions concerning the heights the considered fuzzy sets. The next result states in which way the fulfillment of the above two criteria can be guaranteed.

6.24. Proposition. *Both preorderings $\lesssim_I$ and $\lesssim_L$ satisfy criterion $\boldsymbol{A_4}$ on $\mathcal{S} = \mathcal{F}_H(X)$, where $\lesssim_I$ even satisfies $\boldsymbol{A'_4}$. If the underlying fuzzy equivalence relation E is a fuzzy equality, $\lesssim_L$ fulfills $\boldsymbol{A'_4}$ on $\mathcal{F}_H(X)$ as well. Moreover, the extensionality of all $A \in \mathcal{S} \subseteq \mathcal{F}_H(X)$ is a sufficient condition that $\lesssim_L$ fulfills $\boldsymbol{A'_4}$ on $\mathcal{S}$.*

Proof. Assume that, for two fuzzy sets $A, B \in \mathcal{F}_H(X)$, the condition (6.13) is satisfied. Hence, all elements in $\operatorname{supp}(A)$ and $\operatorname{supp}(B)$ are comparable.

Furthermore, supp(A) and supp(B) must be disjoint and we obtain the following:

$$\forall x \in \text{supp}(B): \; \mu_{\text{LTR}(A)}(x) = \sup_{y \lesssim x} \mu_A(y) = \sup_{y \in \text{supp}(A)} \mu_A(y) = 1$$

This implies that LTR(B) can never exceed LTR(A). The same argument can be applied analogously to RTL and we have proved that $\lesssim_I$ must fulfill criterion $\mathbf{A_4}$. With the well-known representations (cf. Theorem 5.3 and Remark 5.4)

$$\begin{aligned} \text{ATL}(A) &= \text{EXT}(\text{LTR}(A)) \\ \text{ATM}(A) &= \text{EXT}(\text{RTL}(A)) \end{aligned}$$

and the monotonicity of EXT (cf. Corollary 2.55), we can deduce immediately that $\lesssim_L$ satisfies $\mathbf{A_4}$, too.

Taking the fact, that supp(A) and supp(B) are disjoint, and formula (6.13) into account, we see that

$$\begin{aligned} \forall x \in \text{supp}(A): \; \mu_{\text{LTR}(A)}(x) > 0 = \mu_{\text{LTR}(B)}(x) \\ \forall x \in \text{supp}(B): \; \mu_{\text{RTL}(B)}(x) > 0 = \mu_{\text{RTL}(A)}(x) \end{aligned}$$

which implies that $\lesssim_I$ satisfies $\mathbf{A_4'}$ on $\mathcal{F}_H(X)$. Of course, this also transfers to $\lesssim_L$ which coincides with $\lesssim_I$ as long as A and B are extensional (trivial consequence of Theorem 5.3 and Remark 5.4).

Now assume that E is a fuzzy equality. In order to prove that $\lesssim_L$ fulfills $\mathbf{A_4'}$, it is sufficient to show that there is an $x \in \text{supp}(A)$ such that

$$\mu_A(x) > \mu_{\text{EXT}(B)}(x).$$

If this is the case, the assertion ATL$(A) \supset$ ATL(B) follows from the fact that $\lesssim_I$ fulfills $\mathbf{A_4'}$. For arbitrary $x \in \text{supp}(A)$ and $y \in \text{supp}(B)$, compatibility and separatedness entail

$$x \lnsim u \lnsim v \lnsim y \implies E(x,y) \leq E(u,v) < 1.$$

Since height$(A) = 1$, we can find an $x' \in \text{supp}(A)$ such that

$$\mu_A(x') > E(u,v) \geq \sup\{T(\mu_B(y), E(y,x')) \mid y \in \text{supp}(B)\}$$

which completes the proof. $\square$

Criterion $\mathbf{A_5}$ originally stated the independence of a comparison from the other fuzzy alternatives which have to be ranked. This is trivially fulfilled if only two elements are compared. Therefore, it can be skipped here.

The remaining criteria concern the isotonicity of addition and multiplication of fuzzy quantities. Since this is not so easy to transfer to other domains, we will restrict ourselves to $X = \mathbb{R}$ here. Let us denote the addition of fuzzy quantities with $\oplus$ and the multiplication with $\odot$.

$\mathbf{A_6}$ The following implication is fulfilled for all $A, B, C \in \mathcal{S}$:

$$A \precsim B \implies A \oplus C \precsim B \oplus C$$

$\mathbf{A'_6}$ The following holds for all $A, B, C \in \mathcal{S}\backslash\emptyset$:

$$A \precnsim B \implies A \oplus C \precnsim B \oplus C$$

$\mathbf{A_7}$ For all $A, B, C \in \mathcal{S}$,

$$\{0\} \precsim C \wedge A \precsim B \implies A \odot C \precsim B \odot C$$

Theorem 6.21 readily shows that $\mathbf{A_6}$ holds on $\mathcal{S} = \mathcal{F}(X)$ for the preordering $\lesssim_I$ if it is induced by the ordering of real numbers $\leq$. If all fuzzy sets in $\mathcal{S}$ are extensional, the same is true for $\lesssim_L$ (cf. Corollary 6.22) under the assumption that it is induced by a fuzzy ordering L which directly fuzzifies $\leq$. With slight but obvious modifications, we obtain the same for criterion $\mathbf{A_7}$.

The following counterexample, however, shows that $\mathbf{A'_6}$ is not satisfied for $\lesssim_I$ if we consider $\mathcal{S} = \mathcal{F}_H(X)$. Of course, since $\lesssim_I$ is more specific than the relations $\lesssim_L$, it cannot be fulfilled for the preorderings $\lesssim_L$ either:

$$\begin{aligned}
\mu_A(x) &= \begin{cases} \max(0, 1-x) & \text{if } x \geq 0 \\ \max(0, 1+\frac{x}{2}) & \text{otherwise} \end{cases} \\
\mu_B(x) &= \max(0, 1-|x|) \\
\mu_C(x) &= \begin{cases} \max(0, 1-x) & \text{if } x \geq 0 \\ 1 & \text{otherwise} \end{cases} \\
\mu_{A\oplus C}(x) = \mu_{B\oplus C}(x) &= \begin{cases} \max(0, 1-\frac{x}{2}) & \text{if } x \geq 0 \\ 1 & \text{otherwise} \end{cases}
\end{aligned}$$

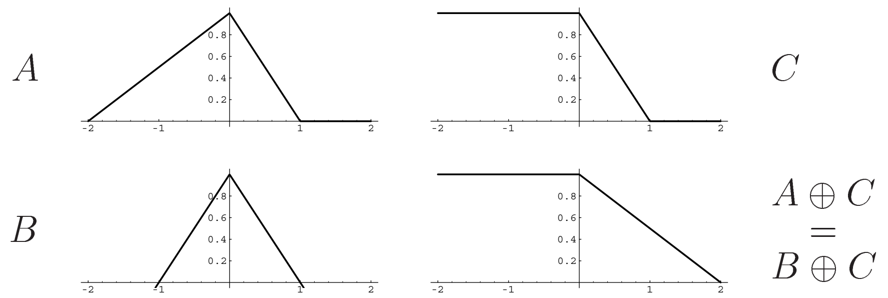


Figure 6.5: A counterexample that criterion $\mathbf{A_6'}$ is not satisfied for $\lesssim_I$.

Summary

1. If an arbitrary ordered domain X is considered, $\lesssim_I$ and $\lesssim_L$ always fulfill $\mathbf{A_1}$–$\mathbf{A_3}$ and $\mathbf{A_5}$ on the entire fuzzy power set $\mathcal{F}(X)$.

2. If $\mathcal{S} = \mathcal{F}_H(X)$, $\mathbf{A_4}$ is always satisfied. The preordering $\lesssim_I$ satisfies criterion $\mathbf{A_4'}$. A relation $\lesssim_L$ fulfills $\mathbf{A_4'}$ if at least either the underlying fuzzy equivalence relation is separated or only extensional fuzzy subsets are considered.

3. Consider the case $X = \mathbb{R}$, where $\lesssim_I$ is induced by the linear ordering of real numbers and $\lesssim_L$ corresponds to some strongly linear ordering L which fuzzifies $\leq$. Then $\lesssim_I$ additionally satisfies $\mathbf{A_6}$ and $\mathbf{A_7}$, where the same is true for $\lesssim_L$ if only extensional fuzzy quantities are considered.

Chapter 7

Conclusion and Outlook

The objective of this dissertation is to show constructive ways of utilizing gradual concepts of orderings for applications in fuzzy systems. By means of three case studies, we have seen that the well-known definitions of fuzzy orderings do have counter-intuitive properties which can be considered as serious obstacles when it concerns applications. The final conclusion was that it contradicts to the nature of vague environments to demand crisp equality in the definition of antisymmetry. This "half-way fuzziness" and the problems arising from it were resolved by replacing the crisp equalities in the definitions of reflexivity and antisymmetry by a similarity relation.

Starting from this key idea of admitting a context-dependent concept of gradual equality, we have discussed several analogies between crisp orderings and the generalized class of fuzzy orderings. In particular, constructions of fuzzy orderings as intersections and Cartesian products have been studied, where it remains an open problem how to define fuzzy orderings of product spaces by lexicographic composition. Moreover, it has been investigated extensively in which way fuzzy orderings can be considered as fuzzifications of crisp orderings including a fundamental result stating that strongly linear fuzzy orderings are uniquely determined as direct fuzzifications of crisp linear orderings.

With the aim of defining ordering-based hedges, which can be particularly useful in a wide scope of applications, hulls of fuzzy orderings and their characterizations have been studied. As an important result, the separability of direct fuzzifications into crisp orderings and fuzzy equivalence relations turned out to transfer to their corresponding hull operators, too.

The last chapter was devoted to methods for defining orderings of fuzzy sets by means of hulls of fuzzy orderings. It is worth to stress again that this approach is not only applicable to convex fuzzy quantities or even smaller subclasses—the new approach can be applied to arbitrary fuzzy subsets of arbitrary, even partially ordered domains. The remaining portion of non-antisymmetry has been characterized including ways to reduce it by hybridization with other ordering methods. Furthermore, connections to other ordering methods have been discussed as well as conditions, under which the extensions of monotonic mappings are also monotonic with respect to the considered orderings of fuzzy sets.

Now, after providing the mathematical apparatus for integrating fuzzy orderings in applied areas, the challenge is to actually perform this integration. As mentioned in Chapter 1, rule interpolation and linguistic approximation are areas where fuzzy orderings could be helpful while ordering-based hedges can provide the basis for reducing the size of fuzzy rule bases while improving interpretability, surveyability, and, therefore, tractability. Moreover, it is, certainly, worth to take a very close look at possible applications in fuzzy decision making.

Symbol Reference

$\emptyset$	empty set	16
$\mathbb{N}$	positive integers	24
$\mathbb{R}$	real numbers	21
$[a, b]$	closed interval	15
$[a, b), (a, b]$	half-open intervals	17
(a, b)	open interval	21
χ_M	characteristic function	16
μ_A	membership function	15
$\mathrm{height}(A)$	height of fuzzy set	16
$\mathrm{supp}(A)$	support of fuzzy set	16
$\mathrm{ceil}(A)$	ceiling of fuzzy set	16
$\mathrm{kern}(A)$	kernel of fuzzy set	16
$\mathrm{gcl}(A, B)$	greatest common level of two fuzzy sets	114
$\mathcal{P}(X)$	power set, i.e., set of crisp subsets	16
$\mathcal{F}(X)$	fuzzy power set, i.e., set of fuzzy subsets	15
$\mathcal{F}_H(X)$	set of fuzzy subsets with height 1	16
$\mathcal{F}_I(X)$	set of convex fuzzy subsets	103
$\mathcal{F}_L(X)$	set of extensional convex fuzzy subsets	103
$\mathcal{F}_N(X)$	set of normal fuzzy subsets	16
$\mathcal{F}_T(X)$	set of fuzzy subsets with non-empty ceiling	16
$[A]_{\underline{\alpha}}$	strict α-cut	17
$[A]_\alpha$	non-strict α-cut	17
$\lfloor A \rfloor_\alpha$	α-base	114

$\lceil A \rceil$	normalization by lifting the ceiling	116
$\cap_T$	intersection of fuzzy sets w.r.t. t-norm T	26
$\cup_S$	union of fuzzy sets w.r.t. t-conorm S	27
$\complement_N$	fuzzy complement w.r.t. negation N	29
$\times_T$	fuzzy Cartesian product w.r.t. t-norm T	36
$A \subseteq B$	subsethood admitting equality; used for both crisp and fuzzy sets	15
$A \subset B$	proper subsethood	79
$T_{\mathbf{M}}$	minimum t-norm	23
$T_{\mathbf{P}}$	product t-norm	23
$T_{\mathbf{L}}$	Łukasiewicz t-norm	23
$T_{\mathbf{W}}$	drastic product	23
$T_{\lambda}^{\mathbf{F}}$	Frank t-norm with parameter λ	23
$S_{\mathbf{M}}$	maximum t-conorm	26
$S_{\mathbf{P}}$	algebraic sum	26
$S_{\mathbf{L}}$	Łukasiewicz t-conorm	26
$S_{\mathbf{W}}$	drastic sum	26
$S_{\lambda}^{\mathbf{F}}$	Frank t-conorm with parameter λ	26
$N_{\mathbf{S}}$	standard negation	27
$N_{\mathbf{I}}$	intuitionistic negation	27
$N_{\mathbf{D}}$	dual intuitionistic negation	27
$x_T^{(n)}$	n-th power w.r.t. t-norm T	24
I_T	S-implication	29
$\vec{T}$	residuum of left-continuous t-norm T	30
$\overleftrightarrow{T}$	biimplication of left-continuous t-norm T	33
$\hat{\varphi}$	extension of mapping φ to fuzzy sets	35
$\hat{*}$	extension of binary operation $*$ to fuzzy sets	122
$\oplus$	addition of fuzzy quantities	126
$\odot$	multiplication of fuzzy quantities	130
R^{-1}	inverse of fuzzy relation	37
R^d	dual of fuzzy relation	37

$\circ_T$	T-composition of fuzzy relations	37
$\sim_E$	kernel relation of fuzzy equivalence relation E	55
$\trianglelefteq_L$	kernel relation of fuzzy ordering L	75
$X_{/\sim}$	factor set w.r.t. equivalence relation $\sim$	73
$X_{/E}$	fuzzy factor set w.r.t. fuzzy equivalence relation E	73
$\langle x \rangle$	equivalence class	74
$H_R(A)$	hull/image w.r.t. fuzzy relation R	41
$\mathrm{EXT}(A)$	extensional hull	45
$\mathrm{ATL}(A)$	hull w.r.t. a fuzzy ordering	88
$\mathrm{ATM}(A)$	hull w.r.t. the inverse of a fuzzy ordering	88
$\mathrm{LTR}(A)$	hull w.r.t. a crisp ordering	89
$\mathrm{RTL}(A)$	hull w.r.t. the inverse of a crisp ordering	91
$\mathrm{CVX}(A)$	convex hull	92
$\mathrm{ECX}(A)$	extensional convex hull	92
SGT, SLS, WIT, ECL, BTW	other ordering-based hedges and connectives	97
$\mathrm{INCL}_T(A, B)$	degree of inclusion of fuzzy sets	53
$\mathrm{SIM}_T(A, B)$	degree of similarity of two fuzzy sets	60
$\leq_I$	interval ordering	100
$\lesssim_I$	preordering of fuzzy sets induced by crisp ordering	102
$\lesssim_L$	preordering of fuzzy sets induced by fuzzy ordering	102
$\cong_I$	symmetric kernel of $\lesssim_I$	103
$\cong_L$	symmetric kernel of $\lesssim_L$	103
$\lesssim'_L, \lesssim''_L$	various generalizations of $\lesssim_L$	114
$\mathcal{L}_{\tilde{T},L}, \mathcal{L}_L$	fuzzy orderings of fuzzy sets	109
$\mathcal{E}_{\tilde{T},L}, \mathcal{E}_L$	fuzzy equivalence relations corresponding to $\mathcal{L}_{\tilde{T},L}$ and $\mathcal{L}_L$, respectively	109

The symbols $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, $\neg$, $\forall$, and $\exists$ have been used solely for Boolean logical expressions. In order to avoid excessive use of parentheses, the following implicit ranking of priority has been assumed by default (1 ... strongest, 5 ... weakest):

1. Relational symbols, such as $=$, $\neq$, $\subseteq$, $\leq$, $<$, etc.
2. Negation $\neg$
3. Conjunction $\wedge$ and disjunction $\vee$
4. Implication $\Rightarrow$ and equivalence $\Leftrightarrow$
5. Quantifiers $\forall$ and $\exists$.

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