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# A NEW APPROACH TO FUZZY ORDERINGS

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ABSTRACT. Previous approaches to fuzzy orderings have a disadvantage that strict antisymmetry criteria lead to analytical properties, which can be unsuitable for real-world applications. In this contribution, a new approach is presented, which overcomes these problems. The key idea is to replace the crisp equality in the definition of reflexivity and antisymmetry by a fuzzy equivalence relation.

## 1. Introduction

Starting in the early seventies, fuzzy relations have been defined, investigated, and applied in many different ways. Two prominent subclasses are concepts of indistinguishability [7, 8, 13, 15] and fuzzy orderings [5, 12, 15].

Indistinguishability relations have turned out to be useful tools for the investigation and interpretation of fuzzy partitions and fuzzy controllers [4, 7, 8, 9]. On the other hand, the utilization of fuzzy orderings in fuzzy control and related fields still seems to be far behind, although there are a lot of reasonable applications, such as ordering-based hedges for reducing the size of rule bases, linguistic approximation, rule interpolation, etc.

First of all, it is near at hand to define a fuzzy concept of ordering just by taking appropriate fuzzifications of the classical three ordering axioms [6, 15].

**DEFINITION 1.1.** A mapping $L\colon X^2 \to [0,1]$ is called a *fuzzy ordering* on the non-empty crisp domain $X$ with respect to a t-norm $T$, for brevity $T$-*ordering*, if and only if the following three axioms are satisfied:

$$\forall x \in X\colon \quad L(x,x) = 1 \quad \text{(reflexivity)},$$

$$\forall x,y \in X\colon \quad x \neq y \Rightarrow T\big(L(x,y), L(y,x)\big) = 0 \quad (T\text{-antisymmetry}),$$

$$\forall x,y,z \in X\colon \quad T\big(L(x,y), L(y,z)\big) \leq L(x,z) \quad (T\text{-transitivity}).$$

One would naturally expect fuzzy orderings to be able to fuzzify crisp linear orderings. Now consider an arbitrary linearly ordered set $X$. A natural re-

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quirement on a sound fuzzification of the original linear ordering would be the following monotonicity:

$$\forall x \in X\colon\ y \preceq z \Longrightarrow L(x,y) \leq L(x,z)\,. \tag{1}$$

Using reflexivity and the above monotonicity, we obtain that, for an arbitrary $x \in X$,

$$\forall y \preceq x\colon\ L(y,x) = 1\,.$$

Linearity of $\preceq$ and $T$-antisymmetry, on the other hand, entail

$$\forall y \not\preceq x\colon\ L(y,x) = 0\,,$$

and we have proven that the crisp ordering $\preceq$ itself is the only fuzzy ordering such that the property (1) is fulfilled.

Moreover, thinking of fuzzy orderings as mathematical models of concepts like *'approximately smaller or equal'* or *'approximately greater or equal'*, one immediately observes that there is an inherent component of indistinguishability. This may entail the demand for bringing fuzzy orderings and indistinguishability together.

In the following, we will show how to define a generalization which also takes indistinguishability into account and, as an important side effect, solves the above fuzzification problem. For proof details we refer to [2] and upcoming publications.

## 2. Linking fuzzy orderings and indistinguishability relations

**Definition 2.1.** A mapping $E\colon X^2 \to [0,1]$ is called a *fuzzy equivalence relation* (indistinguishability or similarity relation) on $X$ with respect to a t-norm $T$, short $T$-*equivalence*, if and only if it has the following properties:

$$\begin{array}{lll} \forall x \in X\colon & E(x,x) = 1 & \text{(reflexivity)}\,,\\ \forall x,y \in X\colon & E(x,y) = E(y,x) & \text{(symmetry)}\,,\\ \forall x,y,z \in X\colon & T\big(E(x,y),E(y,z)\big) \leq E(x,z) & (T\text{-transitivity})\,. \end{array}$$

**Definition 2.2.** A function $L\colon X^2 \to [0,1]$ is called a *fuzzy ordering* on $X$ with respect to a t-norm $T$ and a $T$-equivalence $E$, for brevity $T$-$E$-*ordering*, if and only if it fulfills the following three axioms:

$$\begin{array}{lll} \forall x,y \in X\colon & E(x,y) \leq L(x,y) & (E\text{-reflexivity})\,,\\ \forall x,y \in X\colon & T\big(L(x,y),L(y,x)\big) \leq E(x,y) & (T\text{-}E\text{-antisymmetry})\,,\\ \forall x,y,z \in X\colon & T\big(L(x,y),L(y,z)\big) \leq L(x,z) & (T\text{-transitivity})\,. \end{array}$$

$L$ is called *strongly linear* if, for every pair $(x, y)$, either $L(x, y) = 1$ or $L(y, x) = 1$ holds.

**Proposition 2.3.** *Some basic properties:*

(1) *Every $T$-equivalence is a $T$-$E$-ordering.*

(2) *Every crisp ordering is a fuzzy ordering with respect to any t-norm and the crisp equality.*

(3) *If $L$ is a $T$-$E$-ordering, then its so-called inverse relation $G(x, y) = L(y, x)$ is a $T$-$E$-ordering as well.*

(4) *A $T_1$-$E$-ordering is a $T_2$-$E$-ordering if $T_2$ is weaker than $T_1$.*

(5) *Every $T$-ordering is a fuzzy ordering in the sense of Definition 2.2 with respect to $T$ and the crisp equality.*

At first glance, the new definitions of reflexivity and antisymmetry seem to be stronger than the "classical" ones. However, the last point of the above lemma shows that the new approach is, indeed, a generalization which still contains the existing one given in Definition 1.1.

It is a well-known fact that, in the crisp case, a reflexive and transitive relation (often called preordering) is antisymmetric up to its symmetric kernel which is an equivalence relation. The next result shows that the same holds even in the crisp case — with the difference that there can be several equivalences which can be considered as symmetric kernels.

**Theorem 2.4.** *Consider a reflexive and $T$-transitive binary fuzzy relation $L\colon X^2 \to [0, 1]$ (often called fuzzy preordering). The relation $L$ is a $T$-$E$-ordering for some $T$-equivalence $E$ if and only if, for all $x, y \in X$,*

$$T\big(L(x, y), L(y, x)\big) \leq E(x, y) \leq \min\big(L(x, y), L(y, x)\big)\,.$$

*Moreover, the two bounds are $T$-equivalences themselves.*

The next theorem shows how to define Cartesian products of fuzzy orderings.

**Theorem 2.5.** *Let $X_1, \ldots, X_n$ be crisp sets and let $T$ be an arbitrary t-norm. If $(L_1, \ldots, L_n)$ and $(E_1, \ldots, E_n)$ are families of fuzzy relations such that, for all $i = 1, \ldots, n$,*

(1) *$L_i$ and $E_i$ are binary fuzzy relations on $X_i$,*

(2) *$E_i$ is a $T$-equivalence on $X_i$,*

(3) *$L_i$ is a $T$-$E_i$-ordering on $X_i$.*

*Then the fuzzy relation*

$$\tilde{L}\colon (X_1 \times \cdots \times X_n)^2 \longrightarrow [0, 1]\,,$$
$$\big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\big) \longmapsto \mathop{\mathrm{T}}_{i=1}^{n} L_i(x_i, y_i)$$

*is a fuzzy ordering with respect to $T$ and the fuzzy equivalence relation*

$$\tilde{E}\big((x_1,\ldots,x_n),(y_1,\ldots,y_n)\big) = \mathop{\mathrm{T}}_{i=1}^{n} E_i(x_i,y_i)\,.$$

As already promised, the new model is able to fuzzify crisp linear orderings in an intuitive way such that the monotonicity condition (1) is fulfilled. After a fundamental prerequisite, we can show how.

**Definition 2.6.** Let $\preceq$ be a crisp ordering on $X$ and let $E$ be a fuzzy equivalence relation on $X$. $E$ is called *compatible with* $\preceq$, if and only if the following implication holds for all $x,y,z \in X$:

$$x \preceq y \preceq z \implies E(x,z) \leq \min\big(E(x,y),E(y,z)\big)\,. \tag{2}$$

Compatibility can be interpreted as follows: The two outer elements of a three-element chain are at least as distinguishable as any two inner elements.

**Theorem 2.7.** *Consider a fuzzy relation $L$ on a domain $X$ and a $T$-equivalence $E$. Then the following two statements are equivalent:*

(i) *$L$ is a strongly linear $T$-$E$-ordering.*

(ii) *There exists a linear ordering $\preceq$ the relation $E$ is compatible with, such that $L$ can be represented as follows:*

$$L(x,y) = \begin{cases} 1 & \text{if } x \preceq y\,, \\ E(x,y) & \text{otherwise}\,. \end{cases} \tag{3}$$

**Theorem 2.8.** *Provided that $L$ is a strongly linear $T$-$E$-ordering on $X$ and $\preceq$ is a linear ordering on $X$ such that the representation* (3) *holds, the condition* (1) *is guaranteed to be satisfied.*

Theorems 2.7 and 2.8 state that strongly linear fuzzy orderings are uniquely characterized as fuzzifications of crisp linear orderings, where the fuzzy component can be attributed to a fuzzy equivalence relation.

Example 2.9. In order to demonstrate the expressivity and intuitivity of the new concept of fuzzy orderings, we give the following three examples:

(1) Consider an arbitrary left-continuous t-norm $T$. Then its so-called residual implication, defined as

$$\vec{T}(x,y) = \sup\big\{u \in [0,1] \colon T(u,x) \leq y\big\}\,,$$

is a strongly linear fuzzy ordering on the unit interval with respect to $T$ and the corresponding biimplication

$$\overset{\leftrightarrow}{T}(x,y) = T\big(\vec{T}(x,y),\vec{T}(y,x)\big)\,.$$

Note that $\vec{T}$ never fulfills $T$-antisymmetry, regardless of the choice of $T$.

(2) If we fix a left-continuous t-norm $T$, the fuzzy inclusion relation [1, 6]

$$\mathrm{INCL}_T(A,B) = \inf_{x\in X} \vec{T}\big(A(x), B(x)\big)$$

defines a fuzzy ordering on the fuzzy power set $\mathcal{F}(X)$ with respect to $T$ and

$$\mathrm{SIM}_T(A,B) = \inf_{x\in X} \overset{\leftrightarrow}{T}\big(A(x), B(x)\big) ,$$

which is a well-known mapping for measuring the similarity of fuzzy sets, at least if $T$ is the Łukasiewicz t-norm [11, 14]. It is worth to mention that, for any left-continuous t-norm $T$, $\mathrm{INCL}_T$ is not $T$-antisymmetric.

(3) Theorem 2.7 provides a simple way how to construct strongly linear fuzzy orderings from fuzzy equivalence relations. One easily verifies that

$$E(x,y) = \max\big(1 - |x-y|, 0\big)$$

is a $T_{\mathbf{L}}$-equivalence on the set of real numbers $\mathbb{R}$, which is compatible with the usual ordering $\leq$ on $\mathbb{R}$, where $T_{\mathbf{L}}$ denotes the Łukasiewicz t-norm $\max(x+y-1,0)$. Hence,

$$L(x,y) = \begin{cases} 1 & \text{if } x \leq y\,, \\ \max(1-x+y,0) & \text{otherwise} \end{cases}$$

is a $T_{\mathbf{L}}$-$E$-ordering on the real numbers. More generally, an analogous construction can be carried out for any continuous Archimedean t-norm $T$ via

$$E_T(x,y) = f^{-1}\Big(\min\big(|\varphi(x) - \varphi(y)|,\ f(0)\big)\Big) ,$$

where $f$ denotes an additive generator of $T$ [3] and $\varphi\colon \mathbb{R} \to \mathbb{R}$ is an arbitrary, strictly increasing transformation.

## 3. Applications

In this section, we briefly mention two important applications of the new class of fuzzy orderings. For more detailed investigations, the reader is referred to [2].

### 3.1 Ordering-based linguistic hedges.

Surveyability and interpretability are considered to be fundamental characteristics of fuzzy systems. If, however, rule bases are represented as complete tables, which is quite usual in fuzzy control, the number of rules grows exponentially with the number of input variables, which is a serious restriction in terms of both — surveyability and interpretability. Beside other measures, the integration of ordering-based modifiers (hedges), such as *'at least'*, *'at most'*, *'between'*, etc., for grouping neighboring rules with the same consequents could be a promising approach to keep rule bases compact and interpretable.

Hulls with respect to fuzzy orderings provide a simple way to define such modifiers even with the opportunity to take indistinguishability into account.

**Definition 3.1.** Let $R$ be an arbitrary reflexive and $T$-transitive fuzzy relation on a domain $X$, where the t-norm $T$ is supposed to be left-continuous in the following, and let $A$ be a fuzzy subset of $X$. Then the hull of $A$ with respect to $R$ is defined as

$$H_R(A)(x) = \sup\big\{T\big(A(y), R(y,x)\big) : y \in X\big\}.$$

If $R$ is a $T$-equivalence on $X$, the hull is often called *extensional hull*, which we will denote with the symbol $\mathrm{EXT}(A)$. If $R$ is a fuzzy ordering, the symbol $\mathrm{ATL}(A)$ (*'at least A'*) will be used for $H_R(A)$. Moreover, for the hull with respect to the inverse fuzzy ordering $G(x,y) = L(y,x)$, the symbol $\mathrm{ATM}(A)$ (*'at most A'*) will be used.

**Definition 3.2.** For a fuzzy subset $A$ of a domain $X$, which is equipped with an ordering $\preceq$, we define the following operators (left-to-right and right-to-left continuations, convex hull):

$$\begin{aligned} \mathrm{LTR}(A)(x) &= \sup\big\{A(y) : y \preceq x\big\}, \\ \mathrm{RTL}(A)(x) &= \sup\big\{A(y) : x \preceq y\big\}, \\ \mathrm{CVX}(A)(x) &= \min\big(\mathrm{LTR}(A)(x), \mathrm{RTL}(A)(x)\big). \end{aligned}$$

It is not difficult to observe that LTR and RTL are nothing else than hull operations with respect to the following crisp orderings:

$$\mathrm{LTR}(A) = H_{R_1}(A), \quad \text{where} \quad R_1(x,y) = \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \text{otherwise}, \end{cases}$$

$$\mathrm{RTL}(A) = H_{R_2}(A), \quad \text{where} \quad R_2(x,y) = \begin{cases} 1 & \text{if } x \succeq y, \\ 0 & \text{otherwise}. \end{cases}$$

**DEFINITION 3.3.** Let $L$ be a $T$-$E$-ordering on the domain $X$. Then the operator ECX (extensional convex hull, in direct analogy to CVX) is defined as follows:

$$\mathrm{ECX}(A)(x) = \min\big(\mathrm{ATL}(A)(x),\ \mathrm{ATM}(A)(x)\big)\ .$$

**THEOREM 3.4.** *Provided that $L$ is a strongly linear $T$-$E$-ordering such that the representation (3) holds for some crisp linear ordering $\preceq$, the following equalities hold for every fuzzy subset $A$ of $X$:*

$$\begin{aligned}\mathrm{ATL}(A) &= \mathrm{LTR}\big(\mathrm{EXT}(A)\big) = \mathrm{EXT}\big(\mathrm{LTR}(A)\big)\,,\\ \mathrm{ATM}(A) &= \mathrm{RTL}\big(\mathrm{EXT}(A)\big) = \mathrm{EXT}\big(\mathrm{RTL}(A)\big)\,,\\ \mathrm{ECX}(A) &= \mathrm{CVX}\big(\mathrm{EXT}(A)\big) = \mathrm{EXT}\big(\mathrm{CVX}(A)\big)\,.\end{aligned}$$

### 3.2 A general framework for ordering fuzzy sets.

Orderings/rankings of fuzzy sets play an important role in fuzzy decision analysis but also in linguistic approximation, rule interpolation [10], and many other disciplines. Most previous approaches have in common that they are restricted to certain subclasses of fuzzy sets and that they only work for fuzzy subsets of the real numbers.

The next theorem shows that it is possible to define a preordering of fuzzy sets by means of the above ordering-based hedges even if only a fuzzy ordering, which is not even assumed to be strongly linear, of the given domain is known. Moreover, it provides a unique characterization of non-antisymmetry.

**THEOREM 3.5.** *If $L$ is a linear fuzzy ordering on a domain $X$, then the following binary relation, which is defined on the fuzzy power set $\mathcal{F}(X)$,*

$$A \preceq_L B \iff \mathrm{ATL}(A) \supset \mathrm{ATL}(B) \wedge \mathrm{ATM}(A) \subset \mathrm{ATM}(B)$$

*is reflexive, transitive, and antisymmetric up to the equivalence relation*

$$A \sim_L B \iff \mathrm{ECX}(A) = \mathrm{ECX}(B)\,.$$

**COROLLARY 3.6.** *Since, due to Proposition 2.3, a crisp ordering $\preceq$ can also be regarded as a fuzzy ordering, the same construction as in Theorem 3.5 can be applied and we obtain that*

$$A \preceq_I B \iff \mathrm{LTR}(A) \supset \mathrm{LTR}(B) \wedge \mathrm{RTL}(A) \subset \mathrm{RTL}(B)$$

*is a reflexive and transitive relation on $\mathcal{F}(X)$, which is antisymmetric up to the equivalence relation*

$$A \sim_I B \iff \mathrm{CVX}(A) = \mathrm{CVX}(B)\,.$$

So, we have found a way for defining preorderings of fuzzy sets, which allow us to compare arbitrary fuzzy sets. Unlike other approaches, where the restriction to a special class of fuzzy sets is made at the beginning, this approach can be applied to any kind of fuzzy subsets of a domain for which a crisp or fuzzy ordering is known, with the only restriction that these ordering methods cannot distinguish between fuzzy sets with equal (extensional) convex hulls. In particular, no special assumptions concerning the structure of the space $X$ (e.g., completeness, restriction to real numbers or intervals, etc.) have been made.

For many problems it can be sufficient to treat fuzzy sets with the same (extensional) convex hull as equivalent. If, for what reasons ever, one is interested in a fully antisymmetric ordering, it is sufficient to find orderings of all equivalence classes. Then, by applying lexicographic composition, an ordering of fuzzy sets is obtained, where the coarse comparison is carried out by the above preordering.

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