Relations in Higher-order Fuzzy Logic III

Ulrich Bodenhofer[1], Libor Běhounek[2], and Petr Cintula[2]

[1] Software Competence Center Hagenberg
A-4232 Hagenberg, Austria
ulrich.bodenhofer@scch.at

[2] Institute of Computer Science, Academy of Sciences of the Czech Republic
182 07 Prague 8, Czech Republic
{behounek, cintula}@cs.cas.cz

This contribution is the third and last of a series of talks on relations in higher-order fuzzy logic. The first two [5] have introduced the logical framework (see also [3, 4]) along with a set of basic results that, at first glance, look very familiar. These results, however, have been developed from a much more general basis. Their proofs have been devised independently and resemble closer to proofs known from classical theory than to proofs of results existing in fuzzy set theory.

The purpose of this contribution is to establish links between existing results in fuzzy set theory and the results contained in [5]. Moreover, we provide an interpretation to which extent this new framework really adds value and an educated guess what its potential impact on the further development of theory may be.

Links to Existing Concepts and Results

Graded properties of fuzzy relations

In [5, Section 1], the definitions of the five properties E-extensionality, reflexivity, symmetry, transitivity, and E-antisymmetry (cf. Def. 6) are most crucial. Looking as traditional definitions at first glance, the expressions $\mathrm{Ext}_E(R)$, $\mathrm{Refl}(R)$, $\mathrm{Sym}(R)$, $\mathrm{Trans}(R)$, and $\mathrm{Asym}_E(R)$ are not crisp, but may be true to some degree. An approach in this direction has already been introduced by Gottwald [12, 13] and later on picked up by Jacas and Recasens [16]. These works have in common that they are not based on a general logical framework, but on triangular norms on the unit interval (note, however, that Gottwald uses notations that are inspired by formal logic, similar to the terminology introduced in [5]).

The property of extensionality, to our best knowledge, has only been considered in a crisp way so far [14, 17, 18]. The three properties $\mathrm{Refl}(R)$, $\mathrm{Sym}(R)$, and $\mathrm{Trans}(R)$ appear in [12, 13, 16], at least under the restrictions stated above.

The property $\mathrm{Asym}_E(R)$ is different from the one introduced by Gottwald [12, 13] who starts from Zadeh's definition of antisymmetry [19], but with a general t-norm instead of the minimum. The definition of $\mathrm{Asym}_E(R)$ is inspired by the similarity-based approach to fuzzy orderings (see, e.g., [1, 15] and other publications) and trivially coincides with Gottwald's definition if $E = \mathrm{Id}$. Note that the definition of $\mathrm{Asym}_E(R)$ appears in [16], interestingly, without any reference to the similarity-based approach to fuzzy orderings.

In [5, Section 2], several basic results about relations in higher-order fuzzy logic are provided. The crisp counterparts of the assertions comprised in Theorems 2 and 3 are well-known and can be found in any textbook that contains an adequately deep introduction to fuzzy relations (e.g. [11]). Assertions 1.–3. and 6. from Theorem 2 are also known in the graded framework (see, e.g., [13, Sections 18.4 and 18.6]). The fact that intersections and unions of extensional fuzzy sets are again extensional is also well-known [2, 17], its graded generalization in [5, Theorem 4] constitutes a new finding.

Similarities and partitions

In [5, Section 3], a first step towards a graded theory of equivalence relations and partitions is taken. The degree to which a relation is a similarity, denoted $\mathrm{Sim}(R)$, is defined in the same way as in [13] (again note the difference that Gottwald restricts to the unit interval equipped with a t-norm). The concept of a graded fuzzy partition that is built up on this basis can be considered an entirely new concept. The degree of disjointness $\mathrm{Disj}(\mathcal{X})$ is a straightforward generalization of the disjointness criterion that is well-known from literature [7, 10, 17, 18] (in our notation, being equivalent to $\mathrm{Disj}(\mathcal{X}) = 1$). The degree $\mathrm{Part}(\mathcal{X})$ to which a class of classes $\mathcal{X}$ is a partition is a straightforward generalization of the concept of a T-partition introduced in [7] (being put in a wider context in [10]).

Results like the ones from Theorem 5 are available in [13, Section 18.6, p. 466]. Moreover, crisp counterparts of these assertions and the ones from Theorem 6 occur in literature (see [7, 10, 17, 18] and several others), although the graded framework gives these theorems a rather different flavor. Theorems 7 and 8 closely resemble to some results known from literature [7, 10, 17]. In these papers, however, slightly different ways to construct an equivalence relation from a partition are employed than the relation $\sim_{\mathcal{X}}$ which is only guaranteed to be a fuzzy equivalence relation in the traditional sense (in our framework, being equivalent to $\mathrm{Sim}(\sim_{\mathcal{X}}) = 1$) if $\mathrm{Part}(\mathcal{X}) = 1$ [5, Theorem 8].

Fuzzy orderings and lattice operations

Finally, in [5, Section 4], a graded concept of fuzzy orderings is introduced in line with the similarity-based approach to fuzzy orderings [1, 15]. Gottwald [12, 13] uses the same techniques to define a graded concept of fuzzy partial ordering, but with respect to the crisp equality and not with reference to a fuzzy equivalence relation. Theorem 9 lists results that are well-known in the classical non-graded theory of fuzzy quasiorderings, but new in a graded framework. Assertion 1. is a graded version of the idempotence of the full image with respect to a fuzzy quasiordering [2]. Assertion 2. is a well-known correspondence (see, e.g., [11]). As also known from the classical non-graded theory [2], Assertion 3. is a graded generalization of the fact that the full image of a fuzzy class A with respect to a fuzzy quasiordering R is uniquely represented as the intersection of all R-extensional super-classes of A.

Definitions 10 and 11 can be considered as a starting point towards a general graded theory of fuzzy lattices. The definitions of upper and lower cones, suprema and infima, respectively, appear in the same way as in [6, 8, 9]. Some of the assertions of Theorem 10 are similarly contained in [6].

Conclusion and Outlook

The question remains what kind of value is added by basing a theory of fuzzy relations on the fuzzy class theory as introduced in [4, 5]. First of all, the framework discussed here is well-founded and general. Proofs in this framework are still concise, elegant, and expressive — which is remarkable in light of the fact that all properties of fuzzy relations are graded. Note that Gottwald states in [13, Section 18.6, p. 465] that the development of a full-fledged graded theory of fuzzy equivalence relations and orderings is an open issue. Although the results presented in [5] can only be considered as a good starting point, we strongly believe that this framework has the potential to solve that open issue. The elegance and conciseness of the approach not only allows to generate shorter proofs of many known results in a routine manner. Overcoming the technicality and clumsiness of the classical theory of fuzzy relations may also open the field for discovering completely new results — that is no serious scientific statement based on clear evidence, but a strong belief it is indeed.

Acknowledgements

Ulrich Bodenhofer acknowledges support of the Austrian Government, the State of Upper Austria, and the Johannes Kepler University Linz in the framework of the K*plus* Competence Center Program.

References

1. U. Bodenhofer. A similarity-based generalization of fuzzy orderings preserving the classical axioms. *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems*, 8(5):593–610, 2000.
2. U. Bodenhofer, M. De Cock, and E. E. Kerre. Openings and closures of fuzzy preorderings: Theoretical basics and applications to fuzzy rule-based systems. *Int. J. General Systems*, 32(4):343–360, 2003.
3. L. Běhounek and P. Cintula. From fuzzy logic to fuzzy mathematics: a methodological manifesto. (submitted).
4. L. Běhounek and P. Cintula. Fuzzy class theory. *Fuzzy Sets and Systems*. (to appear).
5. L. Běhounek and P. Cintula. Relations in higher-order fuzzy logic I, II. In *Abstracts 26th Linz Seminar on Fuzzy Set Theory*, Linz, February 2005. (extended abstract).
6. R. Bělohlávek. *Fuzzy Relational Systems. Foundations and Principles*. IFSR Int. Series on Systems Science and Engineering. Kluwer Academic/Plenum Publishers, New York, 2002.
7. B. De Baets and R. Mesiar. T-partitions. *Fuzzy Sets and Systems*, 97:211–223, 1998.
8. M. Demirci. A theory of vague lattices based on many-valued equivalence relation—I: general representation results. *Fuzzy Sets and Systems*. (to appear).
9. M. Demirci. A theory of vague lattices based on many-valued equivalence relation—II: complete lattices. *Fuzzy Sets and Systems*. (to appear).
10. M. Demirci. On many-valued partitions and many-valued equivalence relations. *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems*, 11(2):235–253, 2003.
11. J. Fodor and M. Roubens. *Fuzzy Preference Modelling and Multicriteria Decision Support*. Kluwer Academic Publishers, Dordrecht, 1994.
12. S. Gottwald. *Fuzzy Sets and Fuzzy Logic*. Vieweg, Braunschweig, 1993.

13. S. Gottwald. *A Treatise on Many-Valued Logics*. Studies in Logic and Computation. Research Studies Press, Baldock, 2001.
14. P. Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer Academic Publishers, Dordrecht, 1998.
15. U. Höhle and N. Blanchard. Partial ordering in *L*-underdeterminate sets. *Inform. Sci.*, 35:133–144, 1985.
16. J. Jacas and J. Recasens. Fuzzified properties of fuzzy relations. In *Proc. 9th Int. Conf. on Information Processing and Management of Uncertainty in Knowledge-Based Systems*, volume 1, pages 157–161, Annecy, July 2002.
17. F. Klawonn and J. L. Castro. Similarity in fuzzy reasoning. *Mathware Soft Comput.*, 3(2):197–228, 1995.
18. R. Kruse, J. Gebhardt, and F. Klawonn. *Foundations of Fuzzy Systems*. John Wiley & Sons, New York, 1994.
19. L. A. Zadeh. Similarity relations and fuzzy orderings. *Inform. Sci.*, 3:177–200, 1971.