A Review of Construction and Representation Results for Fuzzy Weak Orders

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Weak orders, i.e. reflexive, transitive, and complete binary relations, are among the most fundamental concepts in preference modeling. It is well-known that weak orders are nothing else but linear orders of equivalence classes, where the corresponding equivalence relation is the symmetric kernel of the weak order. If the underlying set of alternatives X is finite, a weak order can be represented by a single score function [2].

In analogy to the crisp case, fuzzy weak orders are fundamental concepts in fuzzy preference modeling [3, 4, 5]. Given a non-empty set of alternatives X, a fuzzy relation $R : X^2 \to [0, 1]$ is a *fuzzy weak order* if it fulfills the following three axioms for all $x, y, z \in X$ (where T is a left-continuous t-norm):

$R(x, x) = 1$	(reflexivity)
$T(R(x, y), R(y, z)) \leq R(x, z)$	(T-transitivity)
$R(x, y) = 1$ or $R(y, x) = 1$	(strong completeness)

In this contribution, we give an overview of construction and representation results for fuzzy weak orders. This includes both known results and new insights:

(i) Every fuzzy weak order can be represented as a union of a crisp linear order and a fuzzy equivalence relation—which is a full analogue to the crisp case [1]. Based on this discovery, we are able to construct fuzzy weak orders from pseudo-metrics if the t-norm T is continuous Archimedean [1].

(ii) For the case that X is finite, we give a necessary and sufficient condition that a fuzzy weak order is determined only by the degrees to which two consecutive equivalence classes are related to each other.

(iii) Every fuzzy weak order can be represented by score functions [6], but not necessarily by a single one, not even if X is finite [3]. A necessary and sufficient condition for the representability by a single score function is given.

(iv) Fuzzy weak orders can be represented by an embedding to the fuzzy power set $\mathcal{F}(X)$ equipped with the fuzzy inclusion induced by the t-norm T [1].

All these reviews and new results are demonstrated by means of detailed examples.

References

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