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Opening and Closure Operators of Fuzzy Preorderings: Basic Properties and Applications to Fuzzy Rule-Based Systems

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Abstract — The purpose of this paper is twofold. Firstly, a general concept of closedness of fuzzy sets under fuzzy preorderings is proposed and investigated along with the corresponding opening and closure operators. Secondly, the practical impact of this notion is demonstrated by applying it to the analysis of ordering-based modifiers.

Key words — *closedness, fuzzy preordering, fuzzy relation, linguistic modifier.*

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1 Introduction

Images of fuzzy sets under fuzzy relations have been investigated mainly in two contexts: on the one hand, mostly under the term “full image” [4], they can be regarded as very general tools for fuzzy inference, leading to the so-called “compositional rule of inference” [1, 4], which also contains the famous extension principle as a special case [21, 22, 23]. On the other hand, under the term “extensional hull”, the image of a fuzzy set under a fuzzy equivalence relation yields the smallest fuzzy superset which is “closed” under the relation. This closedness property is usually called “extensionality” [14]. The concepts of extensionality and extensional hulls have turned out to be extremely useful, in particular when the analysis and interpretation of fuzzy partitions and controllers is concerned [7, 8, 9, 10].

In the first part of this paper, we would like to generalize the concept of extensionality to arbitrary reflexive and T-transitive fuzzy relations—so-called fuzzy preorderings. Based on this general and powerful notion, smallest closed supersets and largest closed fuzzy subsets will be studied. It will turn out that again two very common concepts of images under fuzzy relations are obtained.

The second part is devoted to a new view on these images of fuzzy sets under fuzzy relations—making use of the results on closedness and the corresponding closure operator, we are able to provide a new framework for defining ordering-based modifiers like “at least”, “at most”, and “between”.

2 Preliminaries

Throughout the whole paper, we will not explicitly distinguish between fuzzy sets and their corresponding membership functions. Consequently, uppercase letters will be used for both synonymously. The set of all fuzzy sets on a domain X will be denoted with $\mathcal{F}(X)$.

For intersecting and unifying fuzzy sets, we will suffice with minimum and maximum:

$$(A\cap B)(x) = \min\big(A(x), B(x)\big)$$
$$(A\cup B)(x) = \max\big(A(x), B(x)\big)$$

In general, triangular norms [11] will be considered as our standard models of conjunction.

Definition 1. A *triangular norm* (*t-norm* for short) is an associative, commutative, and non-decreasing binary operation on the unit interval (i.e. a $[0,1]^2 \to [0,1]$ mapping) which has 1 as neutral element.

In this paper, unless stated otherwise, assume that T denotes a left-continuous triangular norm, i.e. a t-norm whose partial mappings $T(x,.)$ and $T(.,x)$ are left-continuous.

Correspondingly, so-called residual implications are used as the concepts of logical implication. In order to provide the reader with the basic properties of residual implications, we will now briefly recall them. For proofs, the reader is referred to the literature [4, 5].

Definition 2. A mapping $R : [0,1]^2 \to [0,1]$ is called *residual implication (residuum)* of T if and only if the following equivalence is fulfilled for all $x,y,z \in [0,1]$:

$$T(x,y) \le z \iff x \le R(y,z) \tag{1}$$

Lemma 1. *For any left-continuous t-norm T, there exists a unique residuum $\vec{T}$ given as*

$$\vec{T}(x,y) = \sup\{u \in [0,1] \mid T(u,x) \le y\}.$$

Lemma 2. *If T is a left-continuous t-norm, the following holds for all $x,y,z \in [0,1]$:*

1. $x \le y \iff \vec{T}(x,y) = 1$
2. $T\big(\vec{T}(x,y), \vec{T}(y,z)\big) \le \vec{T}(x,z)$
3. $\vec{T}(1,y) = y$
4. $\vec{T}\big(T(x,y),z\big) \le \vec{T}\big(x, \vec{T}(y,z)\big)$
5. $T\big(x, \vec{T}(x,y)\big) \le y$
6. $y \le \vec{T}\big(x, T(x,y)\big)$

Furthermore, $\vec{T}$ is non-increasing and right-continuous in the first argument and non-decreasing and left-continuous in the second argument.

The residual implication can be used to define a logical negation which logically fits to the t-norm and its implication.

Definition 3. The *negation* corresponding to a left-continuous t-norm T is defined as

$$N_T(x) = \vec{T}(x,0).$$

Lemma 3. *N_T is a non-increasing $[0,1] \to [0,1]$ mapping. Moreover, the so-called law of contraposition holds:*

$$\vec{T}(x,y) \le \vec{T}\big(N_T(y), N_T(x)\big)$$

Note that the reverse inequality does not hold in general (unlike the Boolean case, where $p \Rightarrow q$ is equivalent to $\neg q \Rightarrow \neg p$).

In this paper, we will use the negation mainly for the complement of a fuzzy set.

Definition 4. The T*-complement* of a fuzzy set $A \in \mathcal{F}(X)$ is defined as

$$(\complement_T A)(x) = N_T\big(A(x)\big).$$

Lemma 4. *As long as only* min*-intersections and* max*-unions are considered, the so-called De Morgan laws hold:*

$$\complement_T(A \cup B) = \complement_T A \cap \complement_T B$$
$$\complement_T(A \cap B) = \complement_T A \cup \complement_T B$$

Only briefly, we mention the concept of logical equivalence induced by a left-continuous t-norm.

Definition 5. The *biimplication* $\overleftrightarrow{T}$ of T is defined as

$$\overleftrightarrow{T}(x,y) = T\left(\vec{T}(x,y), \vec{T}(y,x)\right).$$

Lemma 5. *The following assertions hold for all* $x,y,z \in [0,1]$*:*

1. $x = y \Longleftrightarrow \overleftrightarrow{T}(x,y) = 1$
2. $\overleftrightarrow{T}(x,y) = \min\left(\vec{T}(x,y), \vec{T}(y,x)\right)$
3. $\overleftrightarrow{T}(x,y) = \overleftrightarrow{T}(y,x)$
4. $T\left(\overleftrightarrow{T}(x,y), \overleftrightarrow{T}(y,z)\right) \leq \overleftrightarrow{T}(x,z)$
5. $\overleftrightarrow{T}(x,y) = \vec{T}\left(\max(x,y), \min(x,y)\right)$

In this paper, we will solely consider *binary fuzzy relations*, i.e. fuzzy sets on a product space $X^2 = X \times X$, where X is an arbitrary crisp set. Let us recall some basics of binary fuzzy relations which will be important in the remaining paper.

Definition 6. A binary fuzzy relation $R : X^2 \to [0,1]$ is called

1. *reflexive* if and only if $\forall x \in X : R(x,x) = 1$,
2. *symmetric* if and only if $\forall x,y \in X : R(x,y) = R(y,x)$,
3. *T-transitive* if and only if $\forall x,y,z \in X : T\left(R(x,y), R(y,z)\right) \leq R(x,z)$,
4. *strongly complete* if and only if $\forall x,y \in X : \max\left(R(x,y), R(y,x)\right) = 1$.

Definition 7. A reflexive and T-transitive fuzzy relation is called *fuzzy preordering* with respect to a t-norm T, short *T-preordering*. A symmetric T-preordering is called *fuzzy equivalence relation* with respect to T, short *T-equivalence*.

Definition 8. Consider an arbitrary fuzzy set $A \in \mathcal{F}(X)$. The *full image* of A under R, denoted $R{\uparrow}A$ and its dual $R{\downarrow}A$ are defined as

$$R{\uparrow}A(x) = \sup\{T\left(A(y), R(y,x)\right) \mid y \in X\},$$
$$R{\downarrow}A(x) = \inf\{\vec{T}\left(R(x,y), A(y)\right) \mid y \in X\}.$$

Note that $R{\uparrow}A$ has sometimes also been called *direct image* [6] or *conditioned fuzzy set* [2], while the names *superdirect image* [6] and @*-operation* [16] have already occurred earlier for $R{\downarrow}A$.

Lemma 6. *The following propositions hold for all $A,B \in \mathcal{F}(X)$ and all binary fuzzy relations $R,S \in \mathcal{F}(X^2)$:*

1. $A \subseteq B \implies R\uparrow A \subseteq R\uparrow B$
2. $A \subseteq B \implies R\downarrow A \subseteq R\downarrow B$
3. $R \subseteq S \implies R\uparrow A \subseteq S\uparrow A$
4. $R \subseteq S \implies R\downarrow A \supseteq S\downarrow A$
5. $R\uparrow(A \cup B) = R\uparrow A \cup R\uparrow B$
6. $R\downarrow(A \cap B) = R\downarrow A \cap R\downarrow B$
7. $(R \cup S)\uparrow A = R\uparrow A \cup S\uparrow A$
8. $(R \cup S)\downarrow A = R\downarrow A \cap S\downarrow A$

Proof. These propositions follow directly from the monotonicity properties of triangular norms and their residual implications (see [4] and [6] for more detailed proofs of (1–5)). □

3 The Basic Concept of Closedness and its Properties

Originally defined for fuzzy equivalence relations under the term "extensionality" [10, 14], we will now define a generalization which does not assume symmetry. Throughout this section, assume that R denotes a fuzzy preordering with respect to some left-continuous t-norm T.

Definition 9. A fuzzy set $A \in \mathcal{F}(X)$ is called *closed* with respect to R, for brevity *R-closed*, if and only if, for all $x,y \in X$,

$$T\big(A(x), R(x,y)\big) \leq A(y).$$

In words, the meaning of closedness is that, for any element x of A, also all y are contained in A which are in relation to x.

Example 1. Let us briefly mention a few simple examples which demonstrate the variety of properties that can be expressed by means of closedness.

1. The universe X and the empty set $\emptyset$ are both closed with respect to any fuzzy preordering on X.
2. A crisp set is closed with respect to a crisp equivalence relation if and only if it can be represented as the union of equivalence classes.
3. A crisp set is closed with respect to a crisp ordering if and only if it is an up-set.
4. A fuzzy set is closed with respect to a crisp ordering $\preceq$ if and only if its membership function is non-decreasing with respect to $\preceq$.

5. If a fuzzy equivalence relation is considered, closedness is equivalent to extensionality [10, 14].

As immediate consequences of the residuation principle (1), we can derive equivalent formulations of R-closedness, which will be helpful later.

Lemma 7. *For any fuzzy set $A \in \mathcal{F}(X)$, R-closedness is equivalent to each of the following two propositions:*

$$\forall x,y \in X : R(x,y) \leq \vec{T}\big(A(x),A(y)\big) \tag{2}$$

$$\forall x,y \in X : A(x) \leq \vec{T}\big(R(x,y),A(y)\big) \tag{3}$$

If R is, in addition, symmetric, A is R-closed if and only if the following inequality holds:

$$\forall x,y \in X : \; R(x,y) \leq \overleftrightarrow{T}\big(A(x),A(y)\big) \tag{4}$$

Proof. The equivalence of R-closedness to Formulae (2) and (3) follows directly from the definition of residual implications.

On the other hand, if we swap x and y in the definition of R-closedness, we obtain

$$T\big(A(y),R(y,x)\big) \leq A(x)$$

which is, due to (2), equivalent to

$$R(y,x) \leq \vec{T}\big(A(y),A(x)\big). \tag{5}$$

If we assume that R is symmetric and taking (2) and (5) into account, we obtain

$$R(x,y) \leq \min\big(\vec{T}(A(x),A(y)),\vec{T}(A(y),A(x))\big) = \overleftrightarrow{T}\big(A(x),A(y)\big).$$

The opposite direction is, i.e. that (4) implies R-closedness, is trivial if we consider (2) and the definition of the biimplication. □

In particular, (2) has a trivial consequence we will need very often in the following.

Corollary 1. *Let Q be another T-preordering. If a fuzzy set A is R-closed and $Q \subseteq R$, then A is also Q-closed.*

The next result clarifies in which way closedness is preserved for finite and infinite unions and intersections (with respect to max and min, respectively).

Lemma 8. *For any family of R-closed fuzzy sets $(A_i)_{i\in I}$, the fuzzy sets defined by*

$$\sup_{i\in I} A_i(x) \quad \textit{and} \quad \inf_{i\in I} A_i(x)$$

are also R-closed. If the index set I is finite, the same holds even if T is not left-continuous.

Proof. For arbitrary $x,y \in X$, we know that

$$T\big(A_i(x), R(x,y)\big) \leq A_i(y)$$

holds for all $i \in I$. Due to the monotonicity of t-norms, R-closedness is then preserved for finite intersections and unions (with respect to minimum and maximum, respectively).The same even holds for infinite intersections if we take the following into account (basic consequence of the monotonicity of t-norms):

$$T(\inf_{i \in I} u_i, v) \leq \inf_{i \in I} T(u_i, v)$$

For infinite unions, left-continuity has to be fulfilled:

$$T\big(\sup_{i \in I} A_i(x), R(x,y)\big) = \sup_{i \in I} T\big(A_i(x), R(x,y)\big) \leq \sup_{i \in I} A_i(y) \qquad \square$$

We will now clarify under which conditions closedness is preserved for the complement of a fuzzy set.

Lemma 9. *Consider an R-closed fuzzy set A. Provided that the relation R is additionally symmetric (i.e. a T-equivalence), the complement $\complement_T A$ is also R-closed (extensional).*

Proof. We know from Lemma 7 that the following holds:

$$R(x,y) \leq \vec{T}\big(A(x), A(y)\big)$$

Taking symmetry and the contrapositive law (cf. Lemma 3) into account, we obtain

$$R(x,y) = R(y,x) \leq \vec{T}\big(A(y), A(x)\big) \leq \vec{T}\big(N_T(A(x)), N_T(A(y))\big).$$

Therefore, $\complement_T A$ must be R-closed as well. $\square$

Nonchalantly speaking, Corollary 1 has shown that the smaller a fuzzy preordering R is, the easier fuzzy sets are R-closed. The next theorem gives a unique characterization of how large a relation R may be such that a given family of fuzzy sets is still R-closed.

Theorem 1. *Consider an arbitrary family of fuzzy sets $\tilde{A} = (A_i)_{i \in I}$. Then*

$$R_{\tilde{A}}(x,y) = \inf\{\vec{T}\big(A_i(x), A_i(y)\big) \mid i \in I\}$$

is a T-preordering which is, in addition, the largest binary fuzzy relation R such that all A_i are R-closed. Furthermore,

$$R'_{\tilde{A}}(x,y) = \inf\{\overleftrightarrow{T}\big(A_i(x), A_i(y)\big) \mid i \in I\}$$

is a T-equivalence and the largest symmetric binary fuzzy relation R such that all A_i are R-closed.

Proof. Reflexivity and T-transitivity of $R_{\tilde{A}}$ follow from basic properties of residual implications (cf. Lemma 2; see [17] for more details). Analogously, reflexivity, symmetry, and T-transitivity of $R'_{\tilde{A}}$ follow from Lemma 5. Closedness and maximality of both relations follow immediately from Lemma 7, Formulae (2) and (4), respectively. $\square$

4 Opening and Closure Operators

Now we can turn to our actual objects of study—opening and closure operators induced by fuzzy preorderings. We will soon see that the two image operators $R\uparrow$ and $R\downarrow$ play a central role; so, let us start to investigate their properties in terms of closedness. Again, we make the convention that R denotes a T-preordering on some fixed domain X.

Proposition 1. *All images $R\uparrow A$ and $R\downarrow A$ are R-closed.*

Proof. For proving that $R\uparrow A$ is R-closed, consider the left-continuity of T and the T-transitivity of R:

$$\begin{aligned}
T(R\uparrow A(x), R(x,y)) &= T\big(\sup\{T(A(z), R(z,x)) \mid z \in X\}, R(x,y)\big) \\
&= \sup\{T\big(T(A(z), R(z,x)), R(x,y)\big) \mid z \in X\} \\
&= \sup\{T\big(A(z), T(R(z,x), R(x,y))\big) \mid z \in X\} \\
&\leq \sup\{T(A(z), R(z,y)) \mid z \in X\} \\
&= R\uparrow A(y)
\end{aligned}$$

If we take T-transitivity of R, the monotonicity properties of $\vec{T}$ and Lemma 2, (4), into account, we obtain

$$\begin{aligned}
R\downarrow A(x) &= \inf\{\vec{T}\big(R(x,z), A(z)\big) \mid z \in X\} \\
&\leq \inf\{\vec{T}\big(T(R(x,y), R(y,z)), A(z)\big) \mid z \in X\} \\
&\leq \inf\{\vec{T}\big(R(x,y), \vec{T}(R(y,z), A(z))\big) \mid z \in X\} \\
&\leq \vec{T}\big(R(x,y), \inf\{\vec{T}\big(R(y,z), A(z)\big) \mid z \in X\}\big) \\
&= \vec{T}\big(R(x,y), R\downarrow A(y)\big)
\end{aligned}$$

which is, by Lemma 7, (3), a sufficient condition for R-closedness. □

Theorem 2. *For any $A \in \mathcal{F}(X)$, $R\uparrow A$ is the smallest R-closed fuzzy superset of A and $R\downarrow A$ is the largest R-closed fuzzy subset.*

Proof. From Proposition 1, we know that $R\uparrow A$ and $R\downarrow A$ are indeed R-closed.

The inclusion properties can be proved as follows:

$$\begin{aligned}
R\downarrow A(x) &= \inf\{\vec{T}\big(R(x,y), A(y)\big) \mid y \in X\} \\
&\leq \vec{T}\big(R(x,x), A(x)\big) = \vec{T}\big(1, A(x)\big) \\
&= A(x) \\
&= T\big(A(x), 1\big) = T\big(A(x), R(x,x)\big) \\
&\leq \sup\{T\big(A(y), R(y,x)\big) \mid y \in X\} \\
&= R\uparrow A(x).
\end{aligned}$$

It remains to show minimality/maximality. Suppose B is an arbitrary R-closed fuzzy superset of A. Then we obtain, for all $x, y \in X$,

$$B(x) \geq T\big(B(y), R(y,x)\big) \geq T\big(A(y), R(y,x)\big)$$

Hence, we can even take the supremum over all y on the right-hand side, i.e.

$$B(x) \geq \sup\{T\big(A(y), R(y,x)\big) \mid y \in X\} = R{\uparrow}A(x),$$

which shows that B must be a fuzzy superset of $R{\uparrow}A$. Since B was chosen arbitrarily, $R{\uparrow}A$ must be the smallest R-closed fuzzy superset.

Now let us consider an arbitrary R-closed fuzzy set C, such that $C \subseteq A$. In a similar way as above, we obtain the following for each $x, y \in X$:

$$C(x) \leq \vec{T}\big(R(x,y), C(y)\big) \leq \vec{T}(R(x,y), A(y))$$

Since this holds for any $x, y \in X$, we can also take the infimum over all y on the right-hand side and the proof of maximality is finished:

$$C(x) \leq \inf\{\vec{T}\big(R(x,y), A(y)\big) \mid y \in X\} = R{\downarrow}A(x)$$ □

According to Theorem 2, it is, therefore, justified to call $R{\uparrow}$ the *closure operator* of R and to call $R{\downarrow}$ the *opening operator* of R.

Corollary 2. *The closure and the opening operator of a T-preordering R can be represented in a dual way:*

$$R{\uparrow}A(x) = \inf\{B(x) \mid B \text{ is an } R\text{-closed fuzzy superset of } A\}$$
$$R{\downarrow}A(x) = \sup\{C(x) \mid C \text{ is an } R\text{-closed fuzzy subset of } A\}$$

Proof. From Theorem 2, we know that any R-closed fuzzy superset of A is a fuzzy superset of $R{\uparrow}A$. Since $R{\uparrow}A$ is an R-closed fuzzy superset of A itself, the representation must hold. The dual representation of $R{\downarrow}A$ can be proved analogously. □

Theorem 2 provides us with the mathematical apparatus for proving several basic properties of closures and openings.

Corollary 3. *The following propositions hold for any $A \in \mathcal{F}(X)$:*

1. *A is R-closed if and only if $A = R{\uparrow}A$.*
2. *A is R-closed if and only if $A = R{\downarrow}A$.*
3. $R{\uparrow}(R{\uparrow}A) = R{\uparrow}A$
4. $R{\downarrow}(R{\downarrow}A) = R{\downarrow}A$
5. $R{\uparrow}(R{\downarrow}A) = R{\downarrow}A$
6. $R{\downarrow}(R{\uparrow}A) = R{\uparrow}A$

Proof. The first two propositions follow directly from Theorem 2. The others are immediate consequences of the first one. □

Items (3) and (4) in Corollary 3 refer to idempotency with respect to composition, i.e. that $R\!\uparrow \circ R\!\uparrow \equiv R\!\uparrow$ and $R\!\downarrow \circ R\!\downarrow \equiv R\!\downarrow$. In order to investigate such algebraic properties a little further, we now formulate a sufficient condition under which the applications of closure and opening operators commute.

Theorem 3. *Given two T-preorderings R_1 and R_2 such that $R_1 \cup R_2$ is T-transitive, the following propositions hold for any $A \in \mathcal{F}(X)$:*

$$(R_1 \cup R_2)\!\uparrow\! A = R_1\!\uparrow\!(R_2\!\uparrow\! A) = R_2\!\uparrow\!(R_1\!\uparrow\! A) = R_1\!\uparrow\! A \cup R_2\!\uparrow\! A$$
$$(R_1 \cup R_2)\!\downarrow\! A = R_1\!\downarrow\!(R_2\!\downarrow\! A) = R_2\!\downarrow\!(R_1\!\downarrow\! A) = R_1\!\downarrow\! A \cap R_2\!\downarrow\! A$$

Proof. $R_1 \cup R_2$ is supposed to be T-transitive. Then the reflexivity of R_1 and R_2 implies that $R_1 \cup R_2$ is a T-preordering, and all the results achieved so far are applicable to $R_1 \cup R_2$ as well.

First of all, $(R_1 \cup R_2)\!\uparrow\! A$ is $R_1 \cup R_2$-closed. Therefore, by Corollary 1, $(R_1 \cup R_2)\!\uparrow\! A$ is R_1-closed and, due to Corollary 3,

$$R_1\!\uparrow\!\big((R_1 \cup R_2)\!\uparrow\! A\big) = (R_1 \cup R_2)\!\uparrow\! A.$$

Since $R_2 \subseteq R_1 \cup R_2$, monotonicity (cf. (1) and (3) of Lemma 6) entails

$$R_1\!\uparrow\!(R_2\!\uparrow\! A) \subseteq R_1\!\uparrow\!\big((R_1 \cup R_2)\!\uparrow\! A\big) = (R_1 \cup R_2)\!\uparrow\! A. \tag{6}$$

The inclusion property (see Theorem 2) and monotonicity (cf. Lemma 6) yield

$$R_1\!\uparrow\! A \subseteq R_1\!\uparrow\!(R_2\!\uparrow\! A), \tag{7}$$
$$R_2\!\uparrow\! A \subseteq R_1\!\uparrow\!(R_2\!\uparrow\! A). \tag{8}$$

Putting (7) and (8) together, we obtain

$$R_1\!\uparrow\! A \cup R_2\!\uparrow\! A \subseteq R_1\!\uparrow\!(R_2\!\uparrow\! A). \tag{9}$$

Since

$$(R_1 \cup R_2)\!\uparrow\! A = R_1\!\uparrow\! A \cup R_2\!\uparrow\! A$$

holds anyway due to Lemma 6, Eq. (9) is equivalent to

$$(R_1 \cup R_2)\!\uparrow\! A \subseteq R_1\!\uparrow\!(R_2\!\uparrow\! A). \tag{10}$$

Then Eq. (6) and (10) together prove that

$$(R_1 \cup R_2)\!\uparrow\! A = R_1\!\uparrow\!(R_2\!\uparrow\! A).$$

The second equality

$$(R_1 \cup R_2)\!\uparrow\! A = R_2\!\uparrow\!(R_1\!\uparrow\! A)$$

follows immediately if we swap R_1 and R_2.

Now let us turn to the second line of equalities. Again, trivially, $(R_1 \cup R_2)\!\downarrow\! A$ is $R_1 \cup R_2$-closed. Hence, due to Corollary 1, $(R_1 \cup R_2)\!\downarrow\! A$ is R_1-closed and, again by Corollary 3,

$$R_1\!\downarrow\!\big((R_1 \cup R_2)\!\downarrow\! A\big) = (R_1 \cup R_2)\!\downarrow\! A.$$

Since $R_2 \subseteq R_1 \cup R_2$, monotonicity (see (2) and (4) of Lemma 6) implies

$$R_1 \downarrow (R_2 \downarrow A) \supseteq R_1 \downarrow \big((R_1 \cup R_2) \downarrow A\big) = (R_1 \cup R_2) \downarrow A. \tag{11}$$

On the other hand, the inclusion property (see Theorem 2) and monotonicity imply

$$\begin{aligned} R_1 \downarrow A &\supseteq R_1 \downarrow (R_2 \downarrow A), \\ R_2 \downarrow A &\supseteq R_1 \downarrow (R_2 \downarrow A). \end{aligned}$$

Joining these two inclusions yields

$$R_1 \downarrow A \cap R_2 \downarrow A \supseteq R_1 \downarrow (R_2 \downarrow A). \tag{12}$$

Since we know from Prop. (8) of Lemma 6 that

$$(R_1 \cup R_2) \downarrow A = R_1 \downarrow A \cap R_2 \downarrow A,$$

the inequalities (11) and (12) imply

$$(R_1 \cup R_2) \downarrow A = R_1 \downarrow (R_2 \downarrow A).$$

The second equality

$$(R_1 \cup R_2) \downarrow A = R_2 \downarrow (R_1 \downarrow A)$$

follows again immediately if we swap R_1 and R_2. □

5 An Application: Ordering-Based Modifiers

Already in their beginning, fuzzy systems were considered as appropriate tools for controlling complex systems and for carrying out complicated decision processes [20]. It is well-known and easy to see that, if rule bases are represented as complete tables, the number of rules grows exponentially with the number of variables—a fact which can be regarded as a serious limitation in terms of surveyability and interpretability.

Almost all fuzzy systems make implicit use of orderings. More specifically, it is quite common to decompose the universe of a system variable into a certain number of fuzzy sets by means of the ordering of the universe—an approach which is often reflected in labels like *"small"*, *"medium"*, or *"large"*. We will now demonstrate by means of a simple example how such ordering information can be used to reduce the size of a rule base while improving expressiveness and interpretability. Consider a typical PD-style fuzzy controller with two inputs e, Δe and one output variable f, where the universes of all these variables are covered by five fuzzy sets labeled *"NB"*, *"NS"*, *"Z"*, *"PS"*, and *"PB"*:

$e \backslash \Delta e$	*NB*	*NS*	*Z*	*PS*	*PB*
NB	*NB*	*NB*	*NB*	*NS*	*Z*
NS	*NB*	*NB*	*NS*	*Z*	*PS*
Z	*NB*	*NS*	*Z*	*PS*	*PB*
PS	*NS*	*Z*	*PS*	*PB*	*PB*
PB	*Z*	*PS*	*PB*	*PB*	*PB*

One possibility to reduce the size of this rule base is to take neighboring rules with the same consequents, such as,

IF	e is *"NB"*	AND	Δe is *"NB"*	THEN	f is *"NB"*
IF	e is *"NS"*	AND	Δe is *"NB"*	THEN	f is *"NB"*
IF	e is *"Z"*	AND	Δe is *"NB"*	THEN	f is *"NB"*

and to replace them by a single rule like the following one[1]:

IF e is *"at most Z"* AND Δe is *"NB"* THEN f is *"NB"*

Of course, there is actually no need to do so in such a simple case. Anyway, grouping neighboring rules by means of expressions, such as, *"at least"*, *"at most"*, or *"between"*, could help to reduce the size of larger rule bases considerably.

In addition, such elements can be useful in rule interpolation. Sometimes, when experts or automatic tuning procedures only provide an incomplete description of a fuzzy rule base, it can still be necessary to obtain a conclusion even if an observation does not match any antecedent in the rule base [12]. Moreover, it is considered as another opportunity for reducing the size of a rule base to store only some representative rules and to interpolate between them [13]. In any case, it is indispensable to have criteria for determining between which rules the interpolation should take place. Beside distance, orderings play a fundamental role in this selection. As an alternative to distance-based methods [13], it is possible to fill the gap between the antecedents of two rules using a fuzzy concept of *"strictly between"*, which leads us to the ordering-based modifiers mentioned above.

The fact remains that we are still lacking a way how to represent such expressions under the presence of fuzziness. In order to have a universal approach which is applicable in a wide variety of practical problems, at least the following two properties should be satisfied:

1. If there is a kind of inherent context of gradual equality in the given environment, ordering-based modifiers should take it into account. Stressing the well-known example of the height of men, this means that a fuzzy set "at least 180cm" should not exclude 179.9cm completely, since both values are almost indistinguishable.

2. Of course, the operators should be applicable to fuzzy sets, too, in order to be able to model expressions like "at least medium".

Usually, an expression like "at least" deeply relies on an underlying concept of ordering. Taking the first of the two above requirements into account , it is, however, not sufficient to consider only crisp concepts of ordering. With the aim to have a vague model of ordering based on an underlying vague concept of equality/equivalence, a generalization of fuzzy orderings has been proposed in [3].

[1] It depends on the underlying inference scheme whether the result is actually the same; we leave this aspect aside for the present paper, since this is not its major concern.

Definition 10. A T-transitive binary fuzzy relation $R \in \mathcal{F}(X^2)$ is called a *fuzzy ordering* on X with respect to a t-norm T and a T-equivalence E, for brevity *T-E-ordering*, if and only if it additionally satisfies the following two axioms:

1. *E-reflexivity:* $\forall x, y \in X : R(x,y) \geq E(x,y)$
2. *T-E-antisymmetry:* $\forall x, y \in X : T\big(R(x,y), R(y,x)\big) \leq E(x,y)$

For more details on this concept of fuzzy orderings, its properties and applications, the reader is referred to [3]. We just mention that, by replacing the fuzzy equivalence relation E by the crisp equality, the well-known definition of fuzzy partial orderings [19] is obtained. Moreover, one easily verifies that this still includes crisp orderings as well.

Now let us start with the problem how to define an operator "at least". If we restrict ourselves to crisp sets and crisp orderings, the following definition seems intuitively correct:

$$x \in \text{“}\textit{at least } M\text{”} \iff \big(\exists y \in X : y \in M \wedge y \preceq x\big).$$

For generalizing this formula to a fuzzy set A and a given T-E-ordering R, two logical concepts have to be fuzzified as well—the conjunction and the existential quantifier. For conjunction, the underlying t-norm T seems to be the ready-made choice. If we take, as usual in t-norm-based predicate logic [5], the supremum as fuzzy substitute for the existential quantifier, the following generalization is obtained:

$$\text{“}\textit{at least } A\text{”}(x) = \sup\{T\big(A(y), R(y,x)\big) \mid y \in X\}$$

Actually, this is nothing else than the full image or closure of A with respect to R:

$$\text{“}\textit{at least } A\text{”} = R{\uparrow}A$$

In order to make our formulas a little shorter and easier to read, we will denote this operator with ATL in the following.

Analogously, it is possible to define an operator *'at most'* just by taking the inverse ordering $R^{-1}(x,y) = R(y,x)$

$$\text{“}\textit{at most } A\text{”}(x) = \sup\{T\big(A(y), R(x,y)\big) \mid y \in X\}$$

which will be denoted ATM in the following. To make notation consistent, let us denote the closure of E—the so-called *extensional hull*—as $\mathrm{EXT}(A) = E{\uparrow}A$.

The question arises what the benefits of the results from Section 3, as promised earlier, are. First of all, and this is neither surprising nor really spectacular, $\mathrm{ATL}(A)$ is R-closed and $\mathrm{ATM}(A)$ is R^{-1}-closed. As an immediate consequence of Corollary 1, $\mathrm{ATL}(A)$ and $\mathrm{ATM}(A)$ are both *extensional*, i.e. E-closed. Moreover, we know from Corollary 3 that both operators are idempotent in the sense that

$$\mathrm{ATL}(\mathrm{ATL}(A)) = \mathrm{ATL}(A),$$
$$\mathrm{ATM}(\mathrm{ATM}(A)) = \mathrm{ATM}(A).$$

We have mentioned above that the T-equivalence-based approach to fuzzy orderings is very much inspired by the typical practical situation that there is a given crisp concept of (mostly linear) ordering, however, with an additional context of gradual equality (like in the height example). We will now study this case in more detail. It will turn out that the results from Section 3 enable us to represent ATL and ATM by closures with respect to the crisp ordering and the fuzzy concept of equality. Before that, let us formalize this typical case in a mathematically exact way.

Definition 11. A T-E-ordering R is called a *direct fuzzification* of a crisp ordering $\preceq$ if and only if it admits the following resolution:

$$R(x,y) = \begin{cases} 1 & \text{if } x \preceq y \\ E(x,y) & \text{otherwise} \end{cases} \tag{13}$$

It is important to mention that strongly complete fuzzy orderings are uniquely characterized as direct fuzzifications of linear orderings [3].

As easy to see from (13), a direct fuzzification of a crisp ordering is the max-union of a crisp ordering and a T-equivalence, which allows us to apply Theorem 3.

Theorem 4. *Let R be a T-E-ordering which is a direct fuzzification of a crisp ordering $\preceq$. Then the following equalities hold*

$$\mathrm{ATL}(A) = \mathrm{EXT}(\mathrm{LTR}(A)) = \mathrm{LTR}(\mathrm{EXT}(A)) = \mathrm{EXT}(A) \cup \mathrm{LTR}(A), \tag{14}$$
$$\mathrm{ATM}(A) = \mathrm{EXT}(\mathrm{RTL}(A)) = \mathrm{RTL}(\mathrm{EXT}(A)) = \mathrm{EXT}(A) \cup \mathrm{RTL}(A), \tag{15}$$

where the operator LTR *denotes the closure with respect to $\preceq$ while* RTL *stands for the closure with respect to the inverse relation of $\preceq$:*

$$\mathrm{LTR}(A)(x) = \sup\{A(y) \mid y \preceq x\}$$
$$\mathrm{RTL}(A)(x) = \sup\{A(y) \mid x \preceq y\}$$

Moreover, $\mathrm{ATL}(A)$ *is the smallest fuzzy superset of A which is extensional and has a non-decreasing membership function. Analogously,* $\mathrm{ATM}(A)$ *is the smallest fuzzy superset of A which is extensional and has a non-increasing membership function.*

Proof. Let us start with the closures induced by the relations $\preceq$ and $\succeq$. For representing $\preceq$ as a fuzzy relation, we consider its *characteristic function*

$$\chi_{\preceq}(x,y) = \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \text{otherwise.} \end{cases}$$

Taking into account that $x \preceq y \Leftrightarrow \chi_{\preceq}(x,y) = 1$, Prop. (2) in Lemma 7 implies that a fuzzy set A is $\preceq$-closed if and only if, for all $x,y \in X$,

$$\chi_{\preceq}(x,y) \leq \vec{T}\big(A(x),A(y)\big).$$

In particular this means that, if $x \preceq y$, the equality $\vec{T}\big(A(x),A(y)\big) = 1$ must hold. Due to (1) in Lemma 2, we obtain that $\preceq$-closedness is equivalent to non-decreasingness of the membership function:

$$x \preceq y \implies A(x) \leq A(y)$$

Analogously, we can show that $\succeq$-closedness corresponds to the non-increasingness of the membership function. Since $\preceq$ is a crisp relation, the following holds:

$$\mathrm{LTR}(A)(x) = \sup\{T\left(A(y), \chi_{\preceq}(y,x)\right) \mid y \in X\} = \sup\{A(y) \mid y \preceq x\}$$

The analogous argument applies to prove the corresponding representation of RTL.

Equality (14) follows directly from Theorem 3 if we consider $R_1 = E$ and $R_2 = \chi_{\preceq}$, while Equality (15) follows in the same way with $R_1 = E$ and $R_2 = \chi_{\succeq}$.

Of course, $\mathrm{ATL}(A)$ is extensional and has a non-decreasing membership function (by Corollary 1, since E and $\preceq$ are both subrelations of R). For proving maximality, suppose that a superset $B \supseteq A$ is extensional and has a non-decreasing membership function. Hence, B is a superset of both $\mathrm{EXT}(A)$ and $\mathrm{LTR}(A)$. Then

$$B \supseteq \mathrm{EXT}(A) \cup \mathrm{LTR}(A) = \mathrm{ATL}(A)$$

must hold, which proves minimality of $\mathrm{ATL}(A)$. Analogous arguments can be applied to prove the minimality of $\mathrm{ATM}(A)$. □

The representations (14) and (15) can be interpreted as commutative diagrams, one of which is shown in Figure 1.

There is one operator we have already mentioned, but not yet dealt with—“between”. In order to introduce such a concept, we have to consider the convexity of fuzzy sets.

Definition 12. Provided that the domain X is equipped with some crisp ordering $\preceq$ (not necessarily linear), a fuzzy set $A \in \mathcal{F}(X)$ is called *convex* (compare with [15, 18]) if and only if, for all $x, y, z \in X$,

$$x \preceq y \preceq z \implies A(y) \geq \min\left(A(x), A(z)\right).$$

Lemma 10. *Any fuzzy set with non-decreasing or non-increasing membership function is convex.*

Proof. Consider a fuzzy set $A \in \mathcal{F}(X)$ with a non-decreasing membership function. Then the following holds for all $x, y, z \in X$:

$$x \preceq y \preceq z \implies A(x) \leq A(y) \leq A(z)$$

Therefore, $A(y) \geq A(x) = \min\left(A(x), A(z)\right)$ must always be fulfilled for an ascending sequence $x \preceq y \preceq z$, and A is guaranteed to be convex. Analogously, the same can be proved for a fuzzy set with non-increasing membership function. □

Therefore, we can conclude, under the assumption that R is a direct fuzzification of some crisp ordering $\preceq$, that $\mathrm{ATL}(A)$, $\mathrm{ATM}(A)$, $\mathrm{LTR}(A)$, and $\mathrm{RTL}(A)$ are convex fuzzy sets for any $A \in \mathcal{F}(X)$.

Lemma 11. *The* min*-intersection of any two convex fuzzy sets is again convex.*

Proof. Assume that A and B are two convex fuzzy sets, i.e.

$$x \preceq y \preceq z \implies A(y) \geq \min\left(A(x), A(z)\right),$$
$$x \preceq y \preceq z \implies B(y) \geq \min\left(B(x), B(z)\right).$$

Taking an arbitrary ascending sequence $x \preceq y \preceq z$, we obtain

$$\begin{aligned}\min\big(A(y),B(y)\big) &\geq \min\Big(\min\big(A(x),A(z)\big),\min\big(B(x),B(z)\big)\Big)\\ &= \min\big(A(x),A(z),B(x),B(z)\big)\\ &= \min\big(A(x),B(x),A(z),B(z)\big)\\ &= \min\Big(\min\big(A(x),B(x)\big),\min\big(A(z),B(z)\big)\Big). \qquad \square\end{aligned}$$

Lemma 12. *Assume that $\preceq$ is an arbitrary, not necessarily linear ordering on a domain X. Then the fuzzy set*

$$\mathrm{CVX}(A) = \mathrm{LTR}(A) \cap \mathrm{RTL}(A)$$

is the smallest convex fuzzy superset of A.

Proof. First of all, $\mathrm{CVX}(A)$ is for sure a convex superset of A, since it is the intersection of two convex fuzzy sets both of which are supersets of A.

Now assume that B is a convex fuzzy superset of A, i.e., for all $x,y,z \in X$,

$$x \preceq y \preceq z \implies B(y) \geq \min\big(B(x),B(z)\big).$$

Since this holds for all chains $x \preceq y \preceq z$, we can even, for a fixed y, take the suprema over all $x \preceq y$ and $z \succeq y$ and the following is obtained:

$$\begin{aligned}B(y) &\geq \min\big(\sup\{B(x) \mid x \preceq y\},\sup\{B(z) \mid y \preceq z\}\big)\\ &= \min\big(\mathrm{LTR}(B)(y),\mathrm{RTL}(B)(y)\big)\\ &\geq \min\big(\mathrm{LTR}(A)(y),\mathrm{RTL}(A)(y)\big)\\ &= \mathrm{CVX}(A)(y)\end{aligned}$$

The fuzzy set B was supposed to be an arbitrary convex fuzzy superset of A; therefore, $\mathrm{CVX}(A)$ must be the smallest convex fuzzy superset of A. $\square$

Theorem 5. *With the assumptions of Theorem 4 and the definition*

$$\mathrm{ECX}(A) = \mathrm{ATL}(A) \cap \mathrm{ATM}(A),$$

the following representation holds:

$$\mathrm{ECX}(A) = \mathrm{EXT}\big(\mathrm{CVX}(A)\big) = \mathrm{CVX}\big(\mathrm{EXT}(A)\big) = \mathrm{EXT}(A) \cup \mathrm{CVX}(A) \tag{16}$$

Furthermore, $\mathrm{ECX}(A)$ *is the smallest fuzzy superset of A which is extensional and convex.*

Proof. Taking into account that, for min and max, the laws of distributivity hold, we obtain the following from Theorem 4:

$$\begin{aligned}\mathrm{ECX}(A)(x) &= \min\big(\mathrm{ATL}(A)(x),\mathrm{ATM}(A)(x)\big)\\ &= \min\big(\max(\mathrm{EXT}(A)(x),\mathrm{LTR}(A)(x)),\\ &\qquad \max(\mathrm{EXT}(A)(x),\mathrm{RTL}(A)(x))\big)\\ &= \max\big(\mathrm{EXT}(A)(x),\min(\mathrm{LTR}(A)(x),\mathrm{RTL}(A)(x))\big)\\ &= \max\big(\mathrm{EXT}(A)(x),\mathrm{CVX}(A)(x)\big)\end{aligned}$$

Using the representations (14) and (15), we immediately obtain from the definition of $\mathrm{CVX}(A)$ that

$$\begin{aligned}\mathrm{ECX}(A) &= \mathrm{ATL}(A) \cap \mathrm{ATM}(A)\\ &= \mathrm{LTR}\big(\mathrm{EXT}(A)\big) \cap \mathrm{RTL}\big(\mathrm{EXT}(A)\big)\\ &= \mathrm{CVX}\big(\mathrm{EXT}(A)\big).\end{aligned}$$

On the other hand, $\mathrm{ECX}(A)$ is an intersection of two convex fuzzy sets and, therefore, convex. Thus, by Lemma 12, $\mathrm{ECX}(A)$ is a fuzzy superset of $\mathrm{CVX}(A)$. Moreover, $\mathrm{ECX}(A)$ is extensional, since it is the intersection of two extensional fuzzy sets (cf. Lemma 8). All together, $\mathrm{ECX}(A)$ is an extensional fuzzy superset of $\mathrm{CVX}(A)$, which implies (cf. Theorem 2)

$$\mathrm{ECX}(A) \supseteq \mathrm{EXT}\big(\mathrm{CVX}(A)\big). \tag{17}$$

Since $A \subseteq \mathrm{CVX}(A)$ always holds, the following is obtained (see (1) of Lemma 6 and Theorem 2):

$$\begin{aligned}\mathrm{EXT}(A) &\subseteq \mathrm{EXT}\big(\mathrm{CVX}(A)\big)\\ \mathrm{CVX}(A) &\subseteq \mathrm{EXT}\big(\mathrm{CVX}(A)\big)\end{aligned}$$

This immediately implies

$$\mathrm{ECX}(A) = \mathrm{EXT}(A) \cup \mathrm{CVX}(A) \subseteq \mathrm{EXT}\big(\mathrm{CVX}(A)\big)$$

which, together with (17), completes the proof of (16).

Now assume that B is an extensional and convex fuzzy superset of A. Since extensionality implies $B \supseteq \mathrm{EXT}(A)$ while convexity implies $B \supseteq \mathrm{CVX}(A)$, we see that

$$B \supseteq \mathrm{CVX}(A) \cup \mathrm{EXT}(A) = \mathrm{ECX}(A)$$

and the minimality of $\mathrm{ECX}(A)$ is proven as well. □

Example 2. Figure 2 shows a simple example of a non-trivial fuzzy set $A \in \mathcal{F}(\mathbb{R})$ and the results which are obtained by applying various operators we have discussed so far.

The relations used for representing these operators are the natural linear ordering of real numbers $\leq$ and the following two fuzzy relations:

$$E(x,y) = \max(1 - |x-y|, 0)$$

$$R(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ \max(1-x+y, 0) & \text{otherwise} \end{cases}$$

One easily verifies that E is, indeed, a $T_{\mathbf{L}}$-equivalence on the real numbers and that R is a $T_{\mathbf{L}}$-E-ordering, which directly fuzzifies the linear ordering of real numbers, where $T_{\mathbf{L}}$ stands for the so-called *Łukasiewicz t-norm*

$$T_{\mathbf{L}}(x,y) = \max(x+y-1, 0).$$

In particular, Fig. 2 demonstrates the commutative diagram shown in Fig. 1 and all the other equalities of (14), (15), and (16).

$$\begin{array}{ccc} A & \xrightarrow{\text{EXT}} & \text{EXT}(A) \\ \text{LTR}\Big\downarrow & & \text{LTR}\Big\downarrow \\ \text{LTR}(A) & \xrightarrow{\text{EXT}} & \text{ATL}(A) \end{array}$$

Figure 1: A commutative diagram depicting the relationships (14) for a given fuzzy set A.

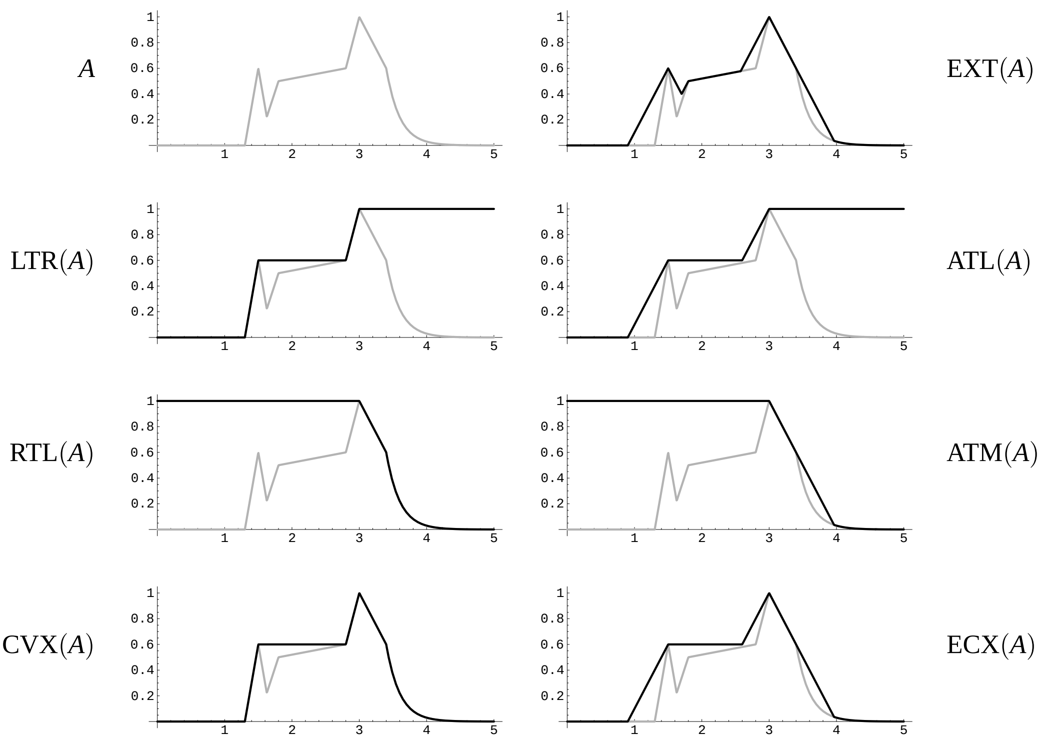

Figure 2: A fuzzy set $A \in \mathcal{F}(\mathbb{R})$ and the results which are obtained when applying various ordering-based operators.

Finally, we can now define an operator representing an inclusive version of “between” with respect to a fuzzy ordering.

Definition 13. Given two fuzzy sets $A,B \in \mathcal{F}(X)$, the binary operator BTW is defined as

$$\mathrm{BTW}(A,B) = \mathrm{ECX}(A \cup B).$$

Easily, we can represent the operator BTW by the operations ATL and ATM only (without any assumptions about specific properties of the underlying T-E-ordering R).

Proposition 2. *The following representation holds for all $A,B \in \mathcal{F}(X)$:*

$$\mathrm{BTW}(A,B) = \big(\mathrm{ATL}(A) \cup \mathrm{ATL}(B)\big) \cap \big(\mathrm{ATM}(A) \cup \mathrm{ATM}(B)\big) \tag{18}$$

Proof. Follows immediately by applying Proposition (5) of Lemma 6:

$$\begin{aligned}\mathrm{BTW}(A,B) &= \mathrm{ECX}(A \cup B)\\ &= \mathrm{ATL}(A \cup B) \cap \mathrm{ATM}(A \cup B)\\ &= \big(\mathrm{ATL}(A) \cup \mathrm{ATL}(B)\big) \cap \big(\mathrm{ATM}(A) \cup \mathrm{ATM}(B)\big)\end{aligned}$$ □

Trivially, $\mathrm{BTW}(A,B)$ is in any case extensional. If R is a direct fuzzification of some crisp ordering $\preceq$, we know by Theorem 5 that $\mathrm{BTW}(A,B)$ is convex as well, a property one would naturally demand of a concept of betweenness. Moreover, the operator is symmetric, i.e.

$$\mathrm{BTW}(A,B) = \mathrm{BTW}(B,A).$$

What remains to be clarified is how an operator “strictly between” can be defined. It seems intuitively clear that “strictly between A and B” should be a subset of $\mathrm{BTW}(A,B)$ which should not include any relevant parts of A and B. The following definition can be considered as the dual of Eq. (18).

Definition 14. The *“strictly between” operator* is a binary connective on $\mathcal{F}(X)$ which is defined as

$$\mathrm{SBT}(A,B) = \complement_T \Big(\big(\mathrm{ATL}(A) \cap \mathrm{ATL}(B)\big) \cup \big(\mathrm{ATM}(A) \cap \mathrm{ATM}(B)\big)\Big).$$

Finally, before we conclude this section with an example, let us clarify some basic properties of the SBT operator.

Proposition 3. *The following representation holds:*

$$\mathrm{SBT}(A,B) = \big(\complement_T \mathrm{ATL}(A) \cup \complement_T \mathrm{ATL}(B)\big) \cap \big(\complement_T \mathrm{ATM}(A) \cup \complement_T \mathrm{ATM}(B)\big) \tag{19}$$

Moreover, the SBT *operator is symmetric and* $\mathrm{SBT}(A,B)$ *is in any case extensional. If R is a direct fuzzification of a crisp ordering $\preceq$,* $\mathrm{SBT}(A,B)$ *is also convex.*

Proof. Eq. (19) follows directly from applying the De Morgan law (Lemma 4) successively.

According to the definition, $\mathrm{SBT}(A,B)$ is given as the complement of a union of intersections of extensional fuzzy sets. Therefore, by Lemmas 8 and 9, $\mathrm{SBT}(A,B)$ must be extensional. Symmetry follows trivially from the symmetry of intersections and unions.

Now assume that R is a direct fuzzification of a crisp ordering $\preceq$. We know that $\mathrm{ATL}(A)$ and $\mathrm{ATL}(B)$ both have non-decreasing membership functions. Since N_T is non-increasing, $\complement_T \mathrm{ATL}(A)$ and $\complement_T \mathrm{ATL}(B)$ both have non-increasing membership functions. The maximum of two non-increasing functions is again non-increasing; therefore,

$$\complement_T \mathrm{ATL}(A) \cup \complement_T \mathrm{ATL}(B)$$

has a non-increasing membership function which implies that it is convex (by Lemma 10). Analogously, one can prove that

$$\complement_T \mathrm{ATM}(A) \cup \complement_T \mathrm{ATM}(B)$$

has a non-decreasing membership function and is, therefore, convex. By Lemma 11, we finally get that

$$\mathrm{SBT}(A,B) = \big(\complement_T \mathrm{ATL}(A) \cup \complement_T \mathrm{ATL}(B)\big) \cap \big(\complement_T \mathrm{ATM}(A) \cup \complement_T \mathrm{ATM}(B)\big)$$

in convex. □

Example 3. Figure 3 shows two fuzzy sets $A,B \in \mathcal{F}(\mathbb{R})$ and the results which are obtained when applying the operators BTW and SBT, where the relations as in Example 2 are used. Figure 4 shows the same, but for the crisp ordering of real numbers (using $T_{\mathbf{L}}$ as underlying t-norm, too).

6 Concluding Remarks

This paper provides a theoretical framework for studying opening and closure operators of fuzzy preorderings. Based on these considerations, we have seen that the results on closure operators have fruitful applications in the construction and analysis of ordering-based modifiers.

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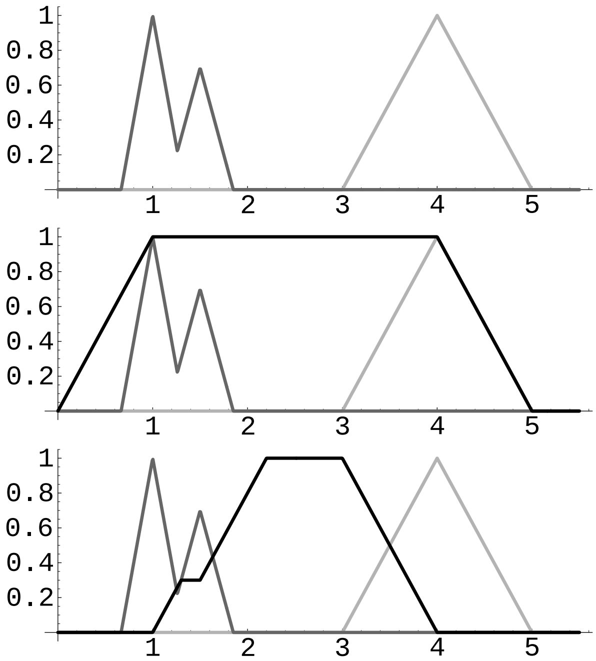

Figure 3: Two fuzzy sets A, B (top) and the results of $\mathrm{BTW}(A, B)$ (middle) and $\mathrm{SBT}(A, B)$ (bottom). The underlying relations are chosen as in Example 2.

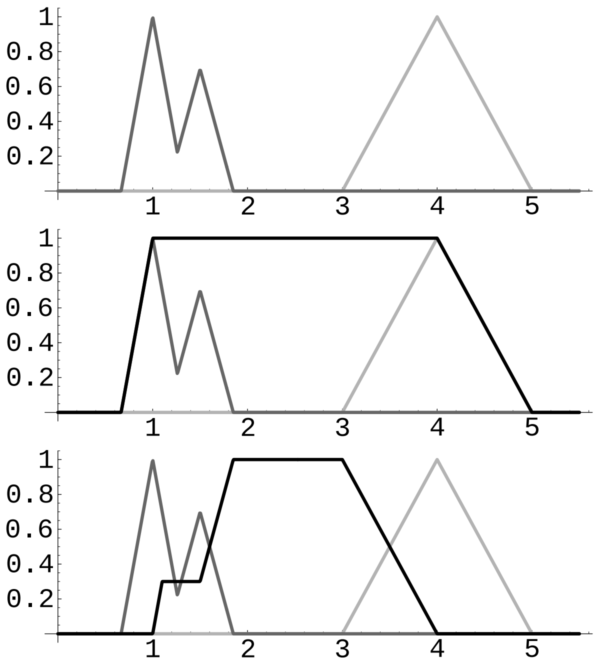

Figure 4: Two fuzzy sets A, B (top) and the results of $\mathrm{BTW}(A, B)$ (middle) and $\mathrm{SBT}(A, B)$ (bottom), using the crisp ordering of real numbers.

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