Strict Fuzzy Orderings in a Similarity-Based Setting

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Abstract

This paper introduces and justifies a similarity-based concept of strict fuzzy orderings and provides constructions how fuzzy orderings can be transformed into strict fuzzy orderings and vice versa. We demonstrate that there is a meaningful correspondence between fuzzy orderings and strict fuzzy orderings. Unlike the classical case, however, we do not obtain a general one-to-one correspondence. We observe that the strongest results are achieved if the underlying t-norm induces a strong negation, which, in particular, includes nilpotent t-norms and the nilpotent minimum.

Keywords: fuzzy equivalence relations, fuzzy orderings, strict fuzzy orderings.

1 Introduction

In the classical case, there is a one-to-one correspondence between partial orderings, i.e. reflexive, antisymmetric, and transitive relations, and strict orderings, i.e. irreflexive and transitive relations. The only trivial component that distinguishes these two concepts is equality. From that point of view, it makes no fundamental difference whether we consider one or the other [20].

Orderings and strict orderings have been studied in the theory of fuzzy relations already as well [11, 17, 18, 23]. Partial fuzzy orderings in the sense of Zadeh [23], however, have severe shortcomings that were finally resolved by replacing the crisp equality by a fuzzy equivalence relation, thereby maintaining the well-known classical fact that orderings are obtained from preorderings by factorization [1–3, 10, 13]. Strict fuzzy orderings based on such a similarity-based setting, however, have not yet been considered so far. This paper aims at filling this gap. We introduce similarity-based strict fuzzy orderings and provide constructions how fuzzy orderings can be transformed into strict fuzzy orderings and vice versa. We will see that, unlike the classical case, the two concepts remain independent to some extent in the sense that there is no general one-to-one correspondence. The reason is for that is twofold: (1) the underlying fuzzy equivalence relation is a much richer structure than the classical equality; (2) the underlying logical operations do not form a Boolean algebra, thus, we do not have the guarantee that all constructions are reversible.

2 Preliminaries

All (fuzzy) relations considered in this paper are binary (fuzzy) relations on a given non-empty domain X. For simplicity, we consider the unit interval $[0, 1]$ as our domain of truth values in this paper. Note that most results, with only minor and obvious modifications, also hold for more general structures [9, 10, 12, 13, 15, 19]. The symbol T denotes a left-continuous t-norm [16]. Correspondingly, $\vec{T}$ denotes the unique residual implication of T. Furthermore, we denote the residual negation of T with $N_T(x) = \vec{T}(x, 0)$. If the residual negation N_T of T is a strong negation (i.e. a continuous, strictly decreasing, and involutive negation), we denote the dual t-conorm (w.r.t. the residual negation N_T) with

$$S_T(x, y) = N_T(T(N_T(x), N_T(y))).$$

In any case, we assume that the reader is familiar with the basic concepts and properties of triangular norms and related operations [11, 16].

Definition 1. A binary fuzzy relation E is called *fuzzy equivalence relation*[1] with respect to T, for brevity *T-equivalence*, if the following three axioms are fulfilled for all $x, y, z \in X$:

1. Reflexivity: $E(x, x) = 1$
2. Symmetry: $E(x, y) = E(y, x)$
3. T-transitivity: $T(E(x, y), E(y, z)) \leq E(x, z)$

Definition 2. A binary fuzzy relation L is called *fuzzy ordering* with respect to T and a T-equivalence E, for brevity *T-E-ordering*, if it fulfills the following three axioms for all $x, y \in X$:

1. E-reflexivity: $E(x, y) \leq L(x, y)$
2. T-E-antisymmetry:
$$T(L(x, y), L(y, x)) \leq E(x, y)$$
3. T-transitivity: $T(L(x, y), L(y, z)) \leq L(x, z)$

Definition 3. A fuzzy relation R is called *strongly complete* if $\max(L(x, y), L(y, x)) = 1$ for all $x, y \in X$ [4, 11, 17]. R is called T-linear if $N_T(L(x, y)) \leq L(y, x)$ for all $x, y \in X$ [4, 13].

Note that strong completeness implies T-linearity, regardless of the choice of T [4]. If N_T is a strong negation, then a fuzzy relation R is T-linear if and only if $S_T(R(x, y), R(y, x)) = 1$ holds for all $x, y \in X$ [4].

3 Strict Fuzzy Orderings

In the crisp case, strict orderings are defined as irreflexive and transitive relations. It is more than obvious how to translate this definition to a fuzzy setting [11, 18]. In order to take the underlying fuzzy equivalence relation into account, we add extensionality.

[1] Note that various diverging names for this class of fuzzy relations appear in literature, like similarity relations, indistinguishability operators, equality relations, and several more [5, 9, 14, 15, 19, 21, 23]

Definition 4. A binary fuzzy relation R is called *strict fuzzy ordering* with respect to T and a T-equivalence E, for brevity *strict T-E-ordering*, if it fulfills the following axioms for all $x, x', y, y', z \in X$:

1. Irreflexivity: $R(x, x) = 0$
2. T-transitivity: $T(R(x, y), R(y, z)) \leq R(x, z)$
3. E-extensionality:
$$T(E(x, x'), E(y, y'), R(x, y)) \leq R(x', y')$$

Note that, under the assumption of T-transitivity, irreflexivity implies T-asymmetry, i.e. that $T(R(x, y), R(y, x)) = 0$ for all $x, y \in X$, where the converse holds only if T does not have zero divisors. In other words, irreflexivity can be replaced equivalently by T-asymmetry if T does not have zero divisors. Furthermore, we can conclude that $T(E(x, y), R(x, y)) = 0$ holds for all $x, y \in X$ and any strict T-E-ordering R.

Example 5. It is a well-known fact that
$$E(x, y) = \max(1 - |x - y|, 0)$$
is a $T_{\mathbf{L}}$-equivalence on $\mathbb{R}$ [7, 21], with $T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0)$ being the Łukasiewicz t-norm. It is easy to show that
$$L(x, y) = \max(\min(1 - x + y, 1), 0)$$
is a strongly complete $T_{\mathbf{L}}$-E-ordering [2, 3] and that
$$R(x, y) = \max(\min(y - x, 1), 0)$$
is a strict $T_{\mathbf{L}}$-E-ordering.

E-extensionality as defined above is nothing else but a straightforward translation of the trivial crisp assertion
$$(x = y \wedge x' = y' \wedge x < y) \rightarrow x' < y'.$$

In case that E is the classical crisp equality, E-extensionality is trivially fulfilled and we end up in the more traditional concept of a strict fuzzy ordering [11, 18]. Conversely, given an irreflexive and T-transitive fuzzy relation, we can make it E-extensional by the following proposition.

Proposition 6. *Let R be an irreflexive and T-transitive fuzzy relation. Then the following fuzzy relation (the* extensional interior of R w.r.t. E*) is a strict T-E-ordering:*

$$\begin{aligned}&\mathrm{Int}_{T,E}[R](x,y)\\&\quad=\inf_{x',y'\in X}\vec{T}(T(E(x,x'),E(y,y')),R(x',y'))\end{aligned}$$

Note that, as the following example suggests, the extensional interior as used in Proposition 6 does not necessarily give a meaningful non-trivial result.

Example 7. The classical strict linear ordering of real numbers $<$ is, of course, an irreflexive and T-transitive fuzzy relation (no matter what t-norm T we choose). Given E from Example 5, we obtain $\mathrm{Int}_{T_{\mathbf{L}},E}[<] = R$ (with R from Example 5). Now let us consider the product t-norm $T_{\mathbf{P}}(x,y) = x \cdot y$. It is well-known that

$$E'(x,y) = \exp(-|x-y|)$$

is a $T_{\mathbf{P}}$-equivalence [7]. However, we obtain that $\mathrm{Int}_{T_{\mathbf{P}},E'}[<]$ is the empty relation, i.e., for all $x,y \in X$,

$$\mathrm{Int}_{T_{\mathbf{P}},E'}[<](x,y) = 0.$$

4 From Fuzzy Orderings to Strict Fuzzy Orderings and Back

In the crisp case, the mutual definability of strict orderings from partial orderings and vice versa is a trivial matter: Given a partial ordering $\leq$, the corresponding strict ordering can be defined as

$$x \leq y \wedge x \neq y$$

or equivalently

$$x \leq y \wedge y \not\leq x.$$

Conversely, given a strict ordering $<$, the relation

$$x < y \vee x = y$$

is a partial ordering. These two constructions are exactly inverse to each other. The question arises whether and how these simple constructions can still be preserved in the more general fuzzy case. The following proposition clarifies the first direction.

Proposition 8. *Consider a T-equivalence E and a T-E-ordering L. Then the following fuzzy relation is a strict T-E-ordering:*

$$\mathrm{Str}_{T,E}[L](x,y) = \min(L(x,y), N_T(L(y,x)))$$

If T does not have zero divisors, the equality $\mathrm{Str}_{T,E}[L](x,y) = \min(L(x,y), N_T(E(y,x)))$ holds additionally.

As a first important property, we obtain that a given T-E-ordering L and the inverse of its induced strict T-E-ordering are disjoint.

Proposition 9. *With the assumptions of Proposition 8, the following equality holds for all $x,y \in X$:*

$$T(L(x,y), \mathrm{Str}_{T,E}[L](y,x)) = 0$$

The definition of $\mathrm{Str}_{T,E}[L]$ is obviously a straightforward translation of the construction $x \leq y \wedge y \not\leq x$ (being equivalent to $x \leq y \wedge x \neq y$ in case that T does not have zero divisors), but it need not be the only possibility to translate this construction to the fuzzy case (e.g. one could use the t-norm T instead of the minimum). Therefore, let us try to investigate whether $\mathrm{Str}_{T,E}[L]$ has some specific properties and, consequently, justifications. We could consider all strict T-E-orderings contained in a T-E-ordering L, but this is not a reasonable assumption. In the crisp case, we would at least assume the following obvious kind of montonicity:

$$\begin{aligned}(x \leq y \wedge y < z) &\rightarrow x < z\\(x < y \wedge y \leq z) &\rightarrow x < z\end{aligned}$$

These properties can be translated into the fuzzy setting in an obvious way.

Definition 10. A fuzzy relation R is called *monotonic* w.r.t. a given T-E-ordering L if and only if the following holds for all $x,y,z \in X$:

$$\begin{aligned}T(L(x,y),R(y,z)) &\leq R(x,z)\\T(R(x,y),L(y,z)) &\leq R(x,z)\end{aligned}$$

The next theorem shows that $\mathrm{Str}_{T,E}[L]$ is the greatest strict T-E-ordering contained in a given T-E-ordering L that fulfills monotonicity with respect to L.

Theorem 11. *Let E be a T-equivalence and let L be a T-E-ordering. Then $\mathrm{Str}_{T,E}[L]$ is the largest strict T-E-ordering that is monotonic w.r.t. L.*

As we are, of course, interested in the most specific information available, i.e. a minimal loss of information, we conclude that $\mathrm{Str}_{T,E}[L]$ is the most appropriate choice how to define a strict T-E-ordering from a given T-E-ordering L. Note that this loss of information can still be severe, as the following example demonstrates.

Example 12. Let us reconsider the $T_{\mathbf{L}}$-equivalence $E(x,y) = \max(1 - |x-y|, 0)$ and the $T_{\mathbf{L}}$-E-ordering $L(x,y) = \max(\min(1-x+y, 1), 0)$. Then we obtain

$$\mathrm{Str}_{T_{\mathbf{L}},E}[L](x,y) = \max(\min(y-x, 1), 0),$$

which is exactly R from Example 5. Now reconsider the $T_{\mathbf{P}}$-equivalence $E'(x,y) = \exp(-|x-y|)$ and the $T_{\mathbf{P}}$-E'-ordering $L'(x,y) = \min(\exp(y-x), 1)$. Then we obtain $\mathrm{Str}_{T_{\mathbf{P}},E'}[L'](x,y) = 0$, i.e. there is no non-trivial strict $T_{\mathbf{P}}$-E'-ordering contained in L' that is monotonic w.r.t. L. Obviously, this is due to the fact that $L(x,y) > 0$ for all $x, y \in \mathbb{R}$ while we have

$$N_T(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

In such a case, therefore, we can never obtain a meaningful strict ordering.

Now let us try to clarify the other direction. The following proposition provides the necessary foundation.

Proposition 13. *Consider a T-equivalence E and a strict T-E-ordering R. Then the following fuzzy relation is a T-E-ordering:*

$$\mathrm{Ref}_{T,E}[R](x,y) = \max(R(x,y), E(x,y))$$

Again the question arises why exactly this choice is appropriate and how it is justified.

Proposition 14. *With the assumptions of Proposition 13, R is monotonic w.r.t. $\mathrm{Ref}_{T,E}[R]$. Moreover, $\mathrm{Ref}_{T,E}[R]$ is the smallest T-E-ordering extending R.*

Now we turn to the question under which conditions the correspondence is one-to-one.

Theorem 15. *Consider a T-equivalence E and a T-E-ordering L. Then the inequality*

$$\mathrm{Ref}_{T,E}[\mathrm{Str}_{T,E}[L]](x,y) \leq L(x,y)$$

holds. The equality

$$\mathrm{Ref}_{T,E}[\mathrm{Str}_{T,E}[L]](x,y) = L(x,y)$$

holds if and only if, for each pair $x, y \in X$, either $T(L(x,y), L(y,x)) = 0$ or $L(x,y) = E(x,y)$ holds.

Theorem 16. *Consider a T-equivalence E and a strict T-E-ordering R. Then the inequality*

$$R(x,y) \leq \mathrm{Str}_{T,E}[\mathrm{Ref}_{T,E}[R]](x,y)$$

holds. If T does not have zero divisors, we even have equality, i.e.

$$R(x,y) = \mathrm{Str}_{T,E}[\mathrm{Ref}_{T,E}[R]](x,y).$$

5 Linearity

Finally, let us approach the question whether linearity (completeness) is preserved by the transformations introduced in the previous section. The concepts of T-linearity and strong completeness as mentioned in Definition 3 are designed for T-E-orderings and are not meaningful for irreflexive relations. Hence, the next definition proposes a straightforward generalization of the well-known property of strict linearity

$$x \neq y \rightarrow (x < y \vee y < x). \tag{1}$$

Definition 17. A fuzzy relation R is called *strictly T-E-linear* (with E being a T-equivalence) if the following inequality holds for all $x, y \in X$:

$$N_T(E(x,y)) \leq \max(R(x,y), R(y,x))$$

Based on this definition, it is possible to prove the following two theorems:

Theorem 18. *Assume we are given a T-equivalence E and a T-E-ordering L. If L is T-linear and fulfills* min*-E-antisymmetry*[2]*, then $\mathrm{Str}_{T,E}[L]$ is strictly T-E-linear.*

[2] i.e. L is a fuzzy ordering in the sense of Bělohlávek [1].

Note that strong completeness is a sufficient condition that T-linearity and min-E-antisymmetry are fulfilled simultaneously.

For the case of a t-norm inducing a strong negation we are able to prove the following stronger results.

Theorem 19. *Suppose we are given a T-equivalence E and a T-E-ordering L and furthermore assume that T induces a strong negation N_T. If L is T-linear, then the following two assertions holds for all $x, y \in X$:*

$$S_T(\mathrm{Str}_{T,E}[L](x,y), \mathrm{Str}_{T,E}[L](y,x)) \geq N_T(E(x,y))$$
$$S_T(\mathrm{Str}_{T,E}[L](x,y), E(x,y), \mathrm{Str}_{T,E}[L](y,x)) = 1$$

The first assertion in Theorem 19 can be understood as a slightly weakened strict T-E-linearity. The second assertion is an important result which is a straightforward generalization of the well-known fact that, in the crisp case, the following holds for any linear ordering $\leq$ (with $<$ being the corresponding strict ordering):

$$x < y \vee x = y \vee y < x$$

Note that this is, of course, an equivalent formulation of (1).

Finally, let us turn to the converse direction.

Theorem 20. *Assume we are given a T-equivalence E and a strict T-E-ordering R. Suppose further that T does not have zero divisors or that T induces a strong negation. If R is strictly T-E-linear, then $\mathrm{Ref}_{T,E}[R]$ is T-linear.*

6 Conclusion

We have introduced and justified a new concept of similarity-based strict fuzzy orderings. Meaningful correspondences between fuzzy orderings and strict fuzzy orderings have been established, but we have not obtained a general one-to-one correspondence. From this point of view, fuzzy orderings and strict fuzzy orderings are not fully equivalent concepts. Hence, the study of both concepts remains interesting and irredundant. Although t-norms without zero divisors give rise to some results that look nice at first glance (see Proposition 8, Theorem 16, and Theorem 20), the examples suggest that this is a rather restrictive and not very intuitive setting. On the other hand, the examples as well as results like Theorems 19 and 20 suggest that t-norms inducing strong negations (in particular, including nilpotent t-norms and the nilpotent minimum) have nice and intuitive properties in this context. This once more confirms the viewpoint that such t-norms are most adequate choices in fuzzy relations theory, fuzzy preference modeling and related fields [4, 6, 8, 22].

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