Linearity Axioms for Fuzzy Orderings: Concepts, Properties, and Difficulties

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The linearity/completeness of orderings is a fundamental property not only in pure mathematics, but also in preference modeling, since it corresponds, in a more general setting, to the absence of incomparability of preference relations.

In the crisp case, an ordering $\preceq$ is called *linear* if and only if, for all $x, y \in X$,

$$(x \preceq y) \vee (y \preceq x). \tag{1}$$

The above axiom is just a simple formulation of a property which has a much deeper meaning in logical and algebraic terms. In particular, there are three essential aspects of relationship between orderings and linear orderings:

(i) Every ordering can be represented as the intersection of linear orderings.

(ii) There is a one-to-one correspondence between linearity and maximality with respect to inclusion, i.e. an ordering is linear if and only if there exists no larger ordering.

(iii) Every ordering can be linearized (Szpilrajn's Theorem [6]).

The fuzzy community has witnessed several approaches to generalizing the concept of completeness to fuzzy relations. This contribution is devoted to a detailed study of the two most common classes of approaches with respect to the three fundamental properties mentioned above, where we consider the general case of fuzzy orderings admitting vague equality [1, 5].

Firstly, many authors have fuzzified (1) by replacing the crisp disjunction with a t-conorm (usually called *S-completeness* [3]):

$$S(R(x, y), R(y, x)) = 1$$

Here the case $S = \max$ plays a specifically important role [2, 3, 8]:

Secondly, it is possible to reformulate (1) such that only the implication (and implicitly the negation) is involved:

$$((x \preceq y) \to \mathbf{0}) \to (y \preceq x). \tag{2}$$

Replacing the crisp implication by a fuzzy implication I yields the second class of approaches [5]. We will refer to this property as *I-completeness* in the following:

$$I(R(x, y), 0) \leq R(y, x)$$

We are going to formulate and prove the following assertions:

a) Property (i) is preserved even for the strongest axiom—max-completeness in the following sense: any T-E-ordering can be represented as the intersection of max-complete relations [7]. The possibility that these relations are T-E-orderings themselves, however, can only be maintained under the condition that the Szpilrajn Theorem holds.

b) The Szpilrajn Theorem is not necessarily fulfilled for the t-conorm-based family of axioms, mostly only under unacceptably strong assumptions [4, 8].

c) If the underlying t-norm T (which is used for defining antisymmetry and transitivity) is left-continuous, then I-completeness with respect to the residual implication of T allows to fulfill the Szpilrajn Theorem [5].

d) In case that the underlying t-norm is not left-continuous, maximality and a corresponding Szpilrajn Theorem do not even make sense.

e) Under the assumptions of c), the following chain of implications holds:

$$\text{max-completeness} \implies \text{maximality} \implies I\text{-completeness}$$

For the special case that the underlying t-norm is nilpotent, there are also correspondences between S-completeness and I-completeness. However, neither S-completeness nor I-completeness have a one-to-one correspondence to maximality.

f) Maximality cannot be expressed by a property which only involves pairs of values (i.e. an expression with only two free variables). More specifically, in the crisp case, the global property of maximality can be characterized by a criterion which is defined locally—for pairs of elements. In the fuzzy case, this characterization does not hold anymore.

We conclude that *it is not possible to formulate a generalized axiom of linearity which can be expressed in a simple form like (1) or (2) and preserves all three fundamental properties.*

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