Linearity Axioms for Fuzzy Orderings: A Formal Review

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Abstract. This contribution is concerned with a review of linearity axioms for fuzzy orderings with respect to three fundamental correspondences from the classical case—linearizability of partial orderings, intersection representation, and one-to-one correspondence between linearity and maximality. We obtain that it is virtually impossible to simultaneously preserve all these three properties in the fuzzy case. If we do not require a one-to-one correspondence between linearity and maximality, however, we obtain that an implication-based definition appears to constitute a sound compromise, in particular, if Łukasiewicz-type logics are considered.

1 Introduction

Orderings are fundamental concepts in mathematics, among which linear orderings play an outstanding role [17]. Beside the context of orderings, in a more general setting, the linearity property also has a great importance in modeling of preferences by relational constructs, since it corresponds to the important property of full comparability (often called *completeness*) or, in other words, absence of incomparability.

Fuzzy relations have been introduced in order to provide more flexible models for expressing relationships [9,10,14–16,21]. The appropriate definition of completeness/linearity, however, is by far not as straightforward as in the classical Boolean case. Several different approaches appear in literature; a systematic formal study with respect to deep logical and algebraic properties, however, has not yet been conducted so far.

The aim of this paper is to investigate three existing definitions of completeness of fuzzy relations in detail. For that purpose, we consider the most fundamental relations for which completeness plays a role—fuzzy orderings—and evaluate the different notions of linearity with respect to fundamental deep results that hold in the crisp case. The final goal is to gain deeper insight into the principles of existing linearity axioms in order to have clear arguments pro and contra their use, not only in connection with fuzzy orderings, but also in more general settings in fuzzy preference modeling.

Note that the given paper is a short version of a longer treatise to be published in [3]. The reader is referred to this upcoming paper for proofs and background details. The paper makes wide use of results on triangular norms and related operations. The reader is referred to the appropriate original literature [9–11,13,18] or the corresponding full paper [3, Section 4].

2 Fundamental Properties in the Crisp Case

In order to clear up notation, let us briefly recall classical orderings (we synonymously use the term *crisp* for Boolean, classical, or non-fuzzy). Throughout the whole paper, assume that the symbol X denotes an arbitrary non-empty set.

Definition 1. A binary relation $\lesssim$ on the set X is called *(partial) ordering* if and only if it fulfills the following three axioms (for all $x, y, z \in X$):

Reflexivity:	$x \lesssim x$
Antisymmetry:	$(x \lesssim y \wedge y \lesssim x) \;\Rightarrow\; x = y$
Transitivity:	$(x \lesssim y \wedge y \lesssim z) \;\Rightarrow\; x \lesssim z$

Definition 2. A binary relation $\diamondsuit$ on X is called *complete* if and only if

$$x \diamondsuit y \vee y \diamondsuit x \tag{1}$$

holds for any pair $x, y \in X$. An ordering fulfilling completeness is called *linear ordering.*

Since this will be important in the following, let us briefly note that (1) is equivalent to

$$x \not\diamondsuit y \;\Rightarrow\; y \diamondsuit x. \tag{2}$$

Completeness is just a simple axiomatization of a property which has a much deeper meaning in logical and algebraic terms. In particular, there are three essential aspects of relationship between (partial) orderings and linear orderings:

[SZP] Any partial ordering can be linearized *(Szpilrajn's Theorem)* [19]: For any partial ordering $\lesssim$, there exists a linear ordering $\preceq$ which extends $\lesssim$ in the sense that, for all $x, y \in X$,

$$x \lesssim y \;\Rightarrow\; x \preceq y. \tag{3}$$

[INT] Any partial ordering can be represented as an intersection of linear orderings [8]: For any partial ordering $\lesssim$, there exists a family of linear orderings $(\preceq_i)_{i\in I}$ such that $\lesssim$ can be represented as (for all $x, y \in X$)

$$x \lesssim y \;\Leftrightarrow\; \bigwedge_{i\in I} x \preceq_i y.$$

[MAX] There is a one-to-one correspondence between linearity and maximality: An ordering $\lesssim$ is linear if and only if there exists no non-trivial extension, i.e. the only linear ordering $\preceq$ fulfilling (3) is $\lesssim$ itself.

These three fundamentally important correspondences will serve as the criteria for evaluating fuzzy linearity/completeness axioms in this paper.

3 Fuzzy Orderings

Binary fuzzy relations were proposed to provide additional freedom for expressing complex preferences that can rarely be modeled in the rigid setting of bivalent logic [9,10,14–16,21]. This is accomplished—as usual in fuzzy set theory—by allowing intermediate degrees of relationship. This paper assumes that the domain of truth values is the common unit interval $[0, 1]$.

This paper addresses the so far most general notion of fuzzy orderings, which—in contrast to earlier approaches—takes an underlying concept of equality/equivalence into account [1,2,12]. This equality/equivalence is modeled by a fuzzy equivalence relation.

Definition 3. A binary fuzzy relation E on X is called *fuzzy equivalence relation* with respect to T, for brevity *T-equivalence*, if and only if the following three axioms are fulfilled for all $x, y, z \in X$:

Reflexivity:	$E(x, x) = 1$
Symmetry:	$E(x, y) = E(y, x)$
T-transitivity:	$T\big(E(x, y), E(y, z)\big) \leq E(x, z)$

Fuzzy relations only fulfilling reflexivity and T-transitivity are called *preorderings* with respect to t-norm T, for short, *T-preorderings.*

Definition 4. Let $L : X^2 \to [0, 1]$ be a binary fuzzy relation. L is called *fuzzy ordering* with respect to T and a T-equivalence E, for brevity *T-E-ordering*, if and only if it is T-transitive and additionally fulfills the following two axioms for all $x, y \in X$:

E-Reflexivity:	$E(x, y) \leq L(x, y)$
T-E-antisymmetry:	$T\big(L(x, y), L(y, x)\big) \leq E(x, y)$

Before the general concept above was introduced, fuzzy orderings were rather commonly understood as T-preorderings that additionally fulfill T-antisymmetry [10,21], i.e., for all $x, y \in [0, 1]$,

$$x \neq y \;\Rightarrow\; T\big(L(x, y), L(y, x)\big) = 0.$$

In order to avoid misunderstandings, let us call this class of fuzzy orderings *T-orderings.* As easy to observe, Definition 4 still accommodates T-orderings if we define E to be the crisp equality. It turned out that basing fuzzy orderings on the crisp equality is too restrictive and practically not feasible. Detailed arguments supporting this viewpoint are elaborated in [1,2,12].

Already in Zadeh's very first paper on fuzzy orderings [21], the fundamental property [SZP] is addressed. If the minimum t-norm $T_{\mathbf{M}}$ is considered for modeling transitivity and antisymmetry (as usual in Zadeh's early works), [SZP] is guaranteed to be satisfied. The proof of this result is simple by using the classical Szpilrajn theorem [19]. A straightforward generalization of this theorem to t-norms without zero divisors was later proved by Gottwald [10]. Although these results seem encouraging at first glance, they do not provide much insight. Nonchalantly speaking, T-orderings, in particular if T does not have zero divisors, are almost crisp concepts. Consequently, [SZP] follows instantly. However, this result relies on the crispness of the concepts under investigation and is by no means applicable if we admit a non-trivial concept of fuzzy equivalence in the sense of Definition 4.

A first serious attempt to investigate [SZP] and [INT] for fuzzy orderings in the sense of Definition 4 was made by Höhle and Blanchard [12]. This paper provides a specific definition of linearity/completeness that has neither become common nor widely known, as it unfortunately remained unknown to the vast majority of the fuzzy set community.

The given paper puts the few dispersed attempts and approaches existing in literature into a common perspective. It considers three major approaches to modeling linearity/completeness—two common in fuzzy preference modeling and the one due to Höhle and Blanchard. All three concepts are checked against the three fundamental properties. In any case, we say that a given concept of linearity/completeness fulfills one of the three fundamental properties if and only if the property is satisfied for all domains X and all T-equivalences E—as a restriction to specific domains or T-equivalences would contradict the generic nature of the fundamental properties in the crisp case. The choice of the logical operators and connectives, however, is crucial for the specific logical framework under investigation. Where possible, characterizations are provided which conditions the logical operators and connectives have to satisfy in order to guarantee that a concept of linearity/completeness fulfills a particular fundamental property.

4 Extensions and the Role of Left-Continuity

All three properties [SZP], [INT], and [MAX] consider extensions of a given fuzzy ordering. This section is devoted to basic definitions and properties that will be essential in the following.

Definition 5. Consider two T-E-orderings L_1 and L_2. We say that L_1 *extends* L_2 if and only if, for all $x, y \in X$, $L_2(x, y) \leq L_1(x, y)$ holds. For brevity, we denote this $L_2 \subseteq L_1$. We call L_1 a *non-trivial extension of* L_2 if there exists at least one pair $(x, y) \in X^2$ for which $L_2(x, y) < L_1(x, y)$ holds, for brevity $L_2 \subset L_1$.

Definition 6. We denote the up-set, the set of elements larger than or equal to (i.e. extending) a given T-E-ordering L, with

$$\mathrm{ext}(L) = \{L' \mid L' \text{ is a } T\text{-}E\text{-ordering and } L \subseteq L'\}.$$

A T-E-ordering L is called *maximal* if and only if it does not have a non-trivial extension, equivalently, $\mathrm{ext}(L) = \{L\}$.

As the next theorem demonstrates, the applicability of Zorn's Lemma in the context of extensions is strictly dependent on the left-continuity of the underlying t-norm.

Theorem 1. *Consider a T-E-ordering L. If T is left-continuous, the set $\mathrm{ext}(L)$ has at least one maximal element.*

Now we turn to the opposite questions, how severe the difficulties are that arise if left-continuity is not satisfied.

Proposition 1. *Provided that the set X has at least two elements and that T is not left-continuous, there exists a T-equivalence E and a linearly ordered sequence of T-E-orderings which does not have a supremum in the set of T-E-orderings on X.*

Proposition 1 particularly implies that we may run into a situation where Zorn's Lemma is not applicable if we consider a t-norm which is not left-continuous. Since, as we will see later, Zorn's Lemma is most often the key to extension theorems à la Szpilrajn, it is unavoidable to *restrict to left-continuous t-norms for the remaining parts of the paper*. It is worth to mention that this is not a serious restriction in logical and practical terms, as t-norms which are not left-continuous fail to fulfill most basic logical properties.

5 Strong Completeness

A simple concept of completeness of fuzzy relations which is common in fuzzy preference modeling [5,6,9] is based on replacing the crisp disjunction in (1) by the maximum t-conorm.

Definition 7. A binary fuzzy relation R on X is called *strongly complete* if and only if the following holds for all $x, y \in X$:

$$\max\big(R(x, y), R(y, x)\big) = 1$$

A unique characterization of T-E-orderings fulfilling strong completeness is available, which we repeat first.

Definition 8. Let $\lesssim$ be a crisp ordering on X and let E be a fuzzy equivalence relation on X. E is called *compatible with* $\lesssim$, if and only if the following implication holds for all $x, y, z \in X$:

$$x \lesssim y \lesssim z \Rightarrow E(x,z) \leq \min\big(E(x,y), E(y,z)\big)$$

Compatibility between a crisp ordering $\lesssim$ and a fuzzy equivalence relation E can be interpreted as follows: The two outer elements of a three-element chain are at least as distinguishable as any two inner elements.

Theorem 2. [1] *Consider a fuzzy relation L on a domain X and a T-equivalence E on the same domain. Then the following two statements are equivalent:*

(i) L is a strongly complete T-E-ordering.
(ii) There exists a linear ordering $\lesssim$ the relation E is compatible with such that L can be represented as follows:

$$L(x,y) = \begin{cases} 1 & \text{if } x \lesssim y \\ E(x,y) & \text{otherwise} \end{cases} \tag{4}$$

As an important consequence of Theorem 2, we obtain that strong completeness implies maximality.

Proposition 2. *For any T-equivalence E, all strongly complete T-E-orderings are maximal.*

Now the question is whether the reverse implication holds, too. The answer, however, is negative, at least if we consider a t-norm which is smaller than the minimum t-norm $T_{\mathbf{M}}$.

Proposition 3. *Assume that $T \neq T_{\mathbf{M}}$, Then, for any set X with at least two elements, there exists a T-equivalence E and a T-E-ordering L for which no strongly complete extension exists.*

Proposition 3 states that [SZP] is not fulfillable for strong completeness if $T \neq T_{\mathbf{M}}$. Trivially, if we have a T-E-ordering L for which no strongly complete extension exists, [INT] cannot hold either, since it is not possible to represent L as the intersection of strongly complete extensions if such extensions do not exist. Moreover, [MAX] does not hold either, since a maximal extension exists for all L (by Theorem 1), even for those for which no strongly complete extension exists.

It remains open so far whether the same problems occur if $T = T_{\mathbf{M}}$ is considered. The following fundamental lemma provides the basis for a full answer.

Lemma 1. *Consider a $T_{\mathbf{M}}$-equivalence E. A T-E-ordering is maximal if and only if it is strongly complete.*

Lemma 1 proves [MAX] for strong completeness for the special case $T = T_{\mathbf{M}}$. As a direct consequence, we obtain that [SZP] holds as well.

Theorem 3 (Szpilrajn Theorem for $T_{\mathbf{M}}$-E-orderings). *Suppose that E is a $T_{\mathbf{M}}$-equivalence. Then any $T_{\mathbf{M}}$-E-ordering has a strongly complete extension.*

The above Szpilrajn-like theorem makes inherent use of Zorn's Lemma, therefore, the result is purely existential.

The question remains whether [INT] can be fulfilled for the case $T = T_{\mathbf{M}}$. The following theorem gives a unique characterization of those $T_{\mathbf{M}}$-E-orderings which can be represented as intersections of strongly complete extensions.

Theorem 4. *Let E be a $T_{\mathbf{M}}$-equivalence and let L be a $T_{\mathbf{M}}$-E-ordering. Then the following two statements are equivalent:*

(i) There exists a family of strongly complete $T_{\mathbf{M}}$-E-orderings $(L_i)_{i\in I}$ such that the following representation holds:

$$L(x, y) = \inf_{i\in I} L_i(x, y)$$

(ii) For all $x, y \in X$, $L(x, y) \in \{E(x, y), 1\}$ holds.

Note that condition (ii) in Theorem 4 directly corresponds to the fact that L is representable like in (4), however, without assuming linearity of the crisp ordering $\lesssim$.

It is easy, for any X, to construct an example of a $T_{\mathbf{M}}$-equivalence E and a $T_{\mathbf{M}}$-E-ordering L such that condition (ii) in Theorem 4 is violated. Hence, [INT] does not hold for strong completeness in the case $T = T_{\mathbf{M}}$ either.

6 T-Linearity

In this section, we consider a type of fuzzy completeness which is based on the idea of generalizing by replacing the Boolean complement by the negation induced by the residual implication of the underlying left-continuous t-norm T [12]. The residual implication of a left-continuous t-norm T is defined as

$$\vec{T}(x, y) = \sup\{z \in [0, 1] \mid T(x, z) \leq y\}.$$

Then the corresponding negation is defined as

$$N_T(x) = \vec{T}(x, 0) = \sup\{z \in [0, 1] \mid T(x, z) = 0\}.$$

Definition 9. A binary fuzzy relation is called *T-linear* if and only if

$$N_T\big(L(x, y)\big) \leq L(y, x)$$

holds for all $x, y \in X$.

Note that $N_T\big(L(x,y)\big) \leq L(y,x)$ is equivalent to

$$\vec{T}\big(N_T\big(L(x,y)\big), L(y,x)\big) = 1,$$

therefore, the direct correspondence to (2) is evident.

The following fundamental theorem provides the basis for proving that [SZP] and [INT] are preserved for T-linearity.

Theorem 5. [12] *Consider a T-equivalence E and a T-E-ordering L. Then, for any pair $(a,b) \in X^2$, there exists a T-linear extension $L_{a,b} \supseteq L$ which fulfills $L(a,b) = L_{a,b}(a,b)$.*

As a trivial consequence, we obtain an appropriate linearization theorem, i.e. a result showing that [SZP] holds for T-linearity.

Corollary 1 (Szpilrajn Theorem for T-linearity). [12] *Given a T-equivalence E, any T-E-ordering has a T-linear extension.*

Moreover, as another consequence of Theorem 5, we can also show that [INT] holds for T-linearity, too.

Corollary 2. [12] *Consider a T-equivalence E. Then, for any T-E-ordering L, there exists a family of T-linear T-E-orderings $(L_i)_{i\in I}$ such that L can be represented as the intersection of all L_i, i.e., for all $x, y \in X$,*

$$L(x,y) = \inf_{i\in I} L_i(x,y).$$

It remains to clarify the correspondence between T-linearity and maximality.

Corollary 3. *Let E be a T-equivalence and L be a T-E-ordering L. If L is maximal, then L is T-linear.*

As we will see next, however, the reverse does not hold in general which implies that the fundamental property [MAX] cannot be preserved for T-linearity.

Proposition 4. *For all domains X with at least two elements, there exists a T-equivalence E and a T-E-ordering L which fulfills T-linearity, but which is not maximal.*

Nonchalantly speaking, this means that T-linearity is, in any case, a property that is “strictly weaker” than maximality. This is particularly true if the t-norm T does not have zero divisors. In such a case, T-linearity only means that, for a fixed pair $(x,y) \in X^2$, $L(x,y) = 0$ implies $L(y,x) = 1$; however, if $\min\big(L(x,y), L(y,x)\big) > 0$, $L(x,y)$ and $L(y,x)$ may take any values from $]0,1]$ without violating T-linearity.

7 S-Completeness

Now we study a generalization of strong completeness which is also well-known in fuzzy preference modeling [9]. It is simply based on replacing the disjunction in (1) by a general t-conorm.

Definition 10. Let S be a t-conorm. A binary fuzzy relation R on X is called S-*complete* if and only if the following holds for all $x, y \in X$:

$$S\big(R(x,y), R(y,x)\big) = 1 \tag{5}$$

In principle, it is possible to consider any t-conorm S. Since we are examining the completeness axioms in the framework of fuzzy orderings, it seems reasonable (and this is also usual even in more general settings in fuzzy preference modeling) to assume a certain structural compatibility between the underlying t-norm T and the t-conorm under consideration. For the remaining section, therefore, assume that (T, S, N) is a de Morgan triple for some strong negation N.

As the first important result, we obtain a full answer to all our questions for the case that T does not have zero divisors.

Lemma 2. *Provided that T does not have zero divisors, S-completeness is equivalent to strong completeness.*

Theorem 6. *Assume that T does not have zero divisors. In the case $T = T_{\mathbf{M}}$, the properties [SZP] and [MAX] hold for S-completeness. If $T \neq T_{\mathbf{M}}$, none of the three fundamental properties holds.*

In particular, this entails that S-completeness does not allow any of the three fundamental properties for strict t-norms—including the important product $T_{\mathbf{P}}$. Now let us approach t-norms with zero divisors. In the first step, we consider t-norms inducing a strong negation.

Lemma 3. *Consider a t-norm T such that N_T is a strong negation. Provided that S is chosen as*

$$S(x,y) = N_T\big(T(N_T(x), N_T(y))\big),$$

then S-completeness is equivalent to T-linearity.

Theorem 7. *Under the assumption that N_T is a strong negation and that we use $N = N_T$, the fundamental properties [SZP] and [INT] hold for S-completeness.*

The class of t-norms inducing a strong negation includes all nilpotent t-norms, most importantly the Łukasiewicz t-norm $T_{\mathbf{L}}$. Moreover, Theorem 7 is also applicable to so-called *nilpotent Zadeh triples* [4], i.e. de Morgan triples

(T_N, S_N, N) where N is a strong negation and where T_N and S_N are defined as follows:

$$T_N(x,y) = \begin{cases} \min(x,y) & \text{if } y > N(x) \\ 0 & \text{otherwise} \end{cases}$$

$$S_N(x,y) = \begin{cases} \max(x,y) & \text{if } x < N(y) \\ 1 & \text{otherwise} \end{cases}$$

This class also comprises the nilpotent minimum $T_{\mathbf{nM}}$ for $N(x) = 1 - x$.

It remains to study what happens if T does have zero divisors and if $N \neq N_T$ ($N = N_T$ can only be fulfilled if T induces a strong negation, anyway). The following theorem provides a sufficient condition for the fulfillment of [SZP] and [INT].

Theorem 8. *Consider a T-equivalence E. If $N \leq N_T$ holds, i.e. $N(x) \leq N_T(x)$ for all $x \in [0,1]$, S-completeness fulfills [SZP] and [INT].*

Now let us study whether $N \leq N_T$ is also a necessary condition for the fulfillment of [SZP] and [INT] by S-completeness.

Lemma 4. *Assume that X has at least two elements. If there is an $\alpha \in]0,1[$ such that $N(\alpha) > N_T(\alpha)$ and additionally $N_T(N_T(\alpha)) = \alpha$ holds, there exists a T-equivalence E and a maximal T-E-ordering L which is not S-complete.*

Theorem 9. *Let X have at least two elements. If there is an $\alpha \in]0,1[$ such that $N(\alpha) > N_T(\alpha)$ and additionally $N_T(N_T(\alpha)) = \alpha$ holds, S-completeness fulfills none of the three fundamental properties.*

The additional requirement $N_T(N_T(\alpha)) = \alpha$ in Lemma 4 and Theorem 9 is not as strong as it might appear at first glance. First of all, if the underlying t-norm T induces a strong negation, this requirement is fulfilled anyway. This also implies that [SZP] and [INT] are fulfilled for Łukasiewicz triples if and only if $N \leq N_T$. The same is true for nilpotent Zadeh triples.

Moreover, if T is a continuous t-norm with zero divisors that is not nilpotent, the conditions of Lemma 4 and Theorem 9, respectively, can be satisfied which implies that none of the three fundamental properties can hold.

We can summarize these findings in the following way:

- If $T = T_{\mathbf{M}}$, S-completeness is equivalent to strong completeness and the fundamental properties [SZP] and [MAX] are fulfilled.
- If T induces a strong negation N_T (including the case of Łukasiewicz triples and nilpotent Zadeh triples), S-completeness fulfills [SZP] and [INT] if and only if $N \leq N_T$.
- For all other continuous t-norms, none of the three fundamental properties can be fulfilled.

Some questions remain open for non-continuous, but left-continuous, t-norms that have zero divisors and do not induce a strong negation. Such t-norms exist of course [13], but they can be considered rather exotic objects of minor practical relevance.

8 Maximality

We are now in the following situation: strong completeness implies maximality, but not vice versa (except for $T = T_{\mathbf{M}}$; cf. Propositions 2 and 3 and Lemma 1); maximality implies T-linearity, but not vice versa (cf. Corollary 3 and Proposition 4). On the one hand, this entails that strong completeness is too strong a property to fulfill any fundamental properties (except for $T = T_{\mathbf{M}}$). On the other hand, T-linearity fulfills [SZP] and [INT], but is too weak a property to fulfill [MAX]. It remains unclear whether there is an appropriate concept of fuzzy linearity/completeness "between" strong completeness and T-linearity which maintains all three fundamental properties. As fulfillment of [MAX] would be nothing else but the equivalence of maximality with this respective property, we can now treat maximality as a concept of fuzzy linearity/completeness in its own right. It is clear then by Theorem 1 that [SZP] is guaranteed to be fulfilled. The only problem remains whether maximality has a reasonable axiomatization, i.e. a simple criterion which allows to check whether a given T-E-ordering is maximal or not.

The following theorem provides a negative result. We obtain that maximality cannot be axiomatized in the usual way by considering pairs or triples of elements only.

Theorem 10. *Consider a domain X with at least four elements and assume that there exists a value $\alpha \in]0,1[$ such that*

$$\alpha = \vec{T}\big(\alpha, T(\alpha,\alpha)\big). \tag{6}$$

Then there exists a T-equivalence E such that maximality of T-E-orderings is not decidable by considering pairs or triples of elements only.

The condition that a value $\alpha \in]0,1[$ fulfilling (6) exists is a merely technical prerequisite for the construction of counterexamples in the proof of Theorem 10. Note that (6) is fulfilled by all continuous t-norms $T \neq T_{\mathbf{M}}$ and all left-continuous t-norms whose induced negation N_T has a fixed point (e.g. including all t-norms T_N).

Theorem 10 states that the fundamental property [MAX] cannot be maintained if we consider completeness axioms like (1) of (2) which both consider pairs of elements only, except for strong completeness in case $T = T_{\mathbf{M}}$. Maximality is a kind of "global" property. In the crisp case, fortunately, maximality remains characterizable by a "local" axiom which only involves pairs of elements. Theorem 10 shows that this nice characterization is lost in the fuzzy case except for the minimum t-norm.

9 Summary and Conclusion

This paper has been concerned with evaluating three concepts of fuzzy linearity/completeness with respect to the three fundamental properties [SZP],

[INT], and [MAX]. The findings can be summarized as follows (see Table 1 for a tabular overview):

Strong completeness: this variant provides reasonable results for the minimum t-norm $T_{\mathbf{M}}$. In this case, [SZP] and [MAX] are fulfilled. A characterization of those $T_{\mathbf{M}}$-E-orderings which admit a representation as intersection of strongly complete $T_{\mathbf{M}}$-E-orderings in the sense of the [INT] property has been given. If $T \neq T_{\mathbf{M}}$, none of the fundamental properties is preserved. Strong completeness, therefore, can only serve as an appropriate fuzzy concept of linearity/completeness if $T = T_{\mathbf{M}}$.

T-linearity: the approach proposed by Höhle and Blanchard provides preservation of [SZP] and [INT] for all left-continuous t-norms. [MAX], however, cannot be satisfied.

S-completeness: in case that T does not have zero divisors, S-completeness coincides with strong completeness (see above). If T is a continuous t-norm that is not nilpotent, none of the three fundamental properties is preserved. In case that T induces a strong negation, [SZP] and [INT] are preserved if and only if $N \leq N_T$. If $N = N_T$, S-completeness and T-linearity are equivalent. The mechanisms underlying these findings are always the results for T-linearity. From that point of view, S-completeness does not provide an essential added value compared to T-linearity.

The first important conclusion that can be drawn from these results if we restrict to commonly used t-norms (continuous t-norms and left-continuous t-norms with strong negation): *the three fundamental properties cannot be preserved simultaneously, no matter which t-norm we choose.*

Secondly, as there is no compact axiomatization of maximality in case $T \neq T_{\mathbf{M}}$, the property [MAX] is not achievable anyway. As this is the property that usually has the least practical relevance compared to [SZP] and [INT], T-linearity constitutes a reasonable compromise that preserves these two properties. However, T-linearity is a very weak, non-intuitive, and poorly expressive concept if T does not induce a strong negation. If T does have a strong negation, T-linearity is not just a compromise, but an almost perfect choice, as T-linearity is equivalent to S-completeness for $N = N_T$. This is not just a nice interpretation, it particularly means that even the two independent fuzzifications of the classical linearity concepts (1) and (2) are equivalent. This result can also be understood as another argument supporting the viewpoint that t-norms inducing strong negations are fundamentally important and beneficial in fuzzy preference modeling [4,7,20].

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Table 1. An overview of the results achieved in this paper

	strong completeness	S-completeness where (T, S, N) is a de Morgan triple	T-linearity
$T_{\mathbf{M}}$	[SZP], [MAX]	[SZP], [MAX]	[SZP], [INT]
other t-norms without zero divisors (e.g. $T_{\mathbf{P}}$)	*none*	*none*	[SZP], [INT]
t-norms inducing a strong negation (e.g. $T_{\mathbf{L}}$, $T_{\mathbf{nM}}$)	*none*	[SZP], [INT], iff $N \leq N_T$	[SZP], [INT]
other continuous t-norms	*none*	*none*	[SZP], [INT]
other left-continuous t-norms	*none*	*???*	[SZP], [INT]

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