T-Transitivity and Domination

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1 Introduction

Aggregation is a fundamental process in decision making and in any other discipline where the fusion of different pieces of information is of vital interest, e.g. in fuzzy querying.

Flexible (fuzzy) querying systems are usually designed not just to give results that match a query exactly, but to give a list of possible answers ranked by their closeness to the query—which is particularly beneficial if no record in the database matches the query in an exact way [14]. The closeness of a single value of a record to the respective value in the query is usually measured using a fuzzy equivalence relation, that is, a reflexive, symmetric and T-transitive fuzzy relation. Recently, a generalization has been proposed [5] which also allows flexible interpretation of ordinal queries (such as "at least" and "at most") by using fuzzy orderings [3]. In any case, if a query consists of at least two expressions that are to be interpreted vaguely, it is necessary to combine the degrees of matching with respect to the different fields in order to obtain an overall degree of matching. Assume that we have a query $(q_1, \ldots, q_n)$, where each $q_i \in X_i$ is a value referring to the i-th field of the query. Given a data record $(x_1, \ldots, x_n)$ such that $x_i \in X_i$ for all $i = 1, \ldots, n$, the overall degree of matching is computed as

$$\tilde{R}\big((q_1, \ldots, q_n), (x_1, \ldots, x_n)\big) = \mathbf{A}\big(R_1(q_1, x_1), \ldots, R_n(q_n, x_n)\big),$$

where each R_i is a T-transitive binary fuzzy relation on X_i which measures the degree to which the value x_i matches the query value q_i.

It appears natural to require that the function $\mathbf{A}$ is an aggregation operator [7, 9, 13] and moreover, it would be desirable that $\tilde{R}$ is still T-transitive in order to have a clear interpretation of the aggregated fuzzy relation $\tilde{R}$. Therefore, it is necessary to study which aggregation operators are able to guarantee that $\tilde{R}$ maintains T-transitivity.

It turns out that the preservation of T-transitivity in aggregating fuzzy relations is closely related to the dominance of an aggregation operator A with respect to the corresponding t-norm T.

Let us recall some basic definitions.

Definition 1. [8] A (binary) operation $\mathbf{A} : [0, 1]^2 \to [0, 1]$ is called an *aggregation operator* if $\mathbf{A}$ is non-decreasing and the equalities $\mathbf{A}(0, 0) = 0$ and $\mathbf{A}(1, 1) = 1$ hold. Moreover, if $\mathbf{A}$ is also associative, symmetric and has 1 as neutral element, then it is called a t-norm.

Definition 2. Consider a binary fuzzy relation R on some universe X and an arbitrary t-norm T. R is called T-transitive if and only if, for all $x, y, z \in X$,

$$T\big(R(x, y), R(y, z)\big) \leq R(x, z). \quad (1)$$

For more details on fuzzy relations, especially fuzzy equivalence relations and fuzzy orderings and their properties, we recommend either original sources as [17, 1, 12], but also [10, 11, 4, 2].

Standard aggregation of fuzzy equivalence relations (fuzzy orderings) preserving T-transitivity is done either be means of T or $T_{\mathbf{M}}(x, y) = \min(x, y)$. Staying in the framework of t-norms, in fact any t-norm T^* dominating T can be applied to preserve T-transitivity, i.e. if R_1, R_2 are two T-transitive, binary relations on a universe X , then also $T^*(R_1, R_2)$ has this property (see [10]). Recall that trivially, for any t-norm T, it holds that T itself and $T_{\mathbf{M}}$ dominate T.

As already mentioned above in several applications, other types of aggregation preserving T-transitivity are required [6]. Especially different weights (degrees of importance) of input fuzzy equivalences (orderings) R_1 and R_2

cannot be properly modeled by aggregation with t-norms, because of the commutativity . Therefore, we have to consider general T-transitivity-preserving aggregation operators.

Note that in the sequel we will deal with the aggregation of two given T-transitive binary fuzzy relations R_1, R_2 acting on the same universe X. Our results can be easily modified for the case of the Cartesian product of T-transitive equivalence relations, as well as to the case of aggregating more than two T-transitive fuzzy relations such that the resulting output fuzzy relation will still be T-transitive.

2 T-Transitivity and Domination

Definition 3. [8] Let $\mathbf{A}, \mathbf{B}$ be two aggregation operators. We say that $\mathbf{A}$ *dominates* $\mathbf{B}$ ($\mathbf{A} \gg \mathbf{B}$), if and only if, for all $x, y, u, v \in [0, 1]$,

$$\mathbf{B}\big(\mathbf{A}(x, y), \mathbf{A}(u, v)\big) \leq \mathbf{A}\big(\mathbf{B}(x, u), \mathbf{B}(y, v)\big). \tag{2}$$

Observe that $\mathbf{A} \gg \mathbf{A}$ if and only if $\mathbf{A}$ is bisymmetric. As already mentioned, for any t-norm T, $T \gg T$ and $T_{\mathbf{M}} \gg T$.

Further on we will denote the class of all aggregation operators $\mathbf{A}$ which dominate a given t-norm T with $\mathcal{D}_T = \{\mathbf{A} \mid \mathbf{A} \gg T\}$.

The following theorem generalizes the result from [10].

Theorem 4. *Let $|X| > 2$. An aggregation operator $\mathbf{A}$ preserves the T-transitivity of fuzzy relations on X if and only if $\mathbf{A} \in \mathcal{D}_T$.*

In the following, we will focus on the characterization of the system $\mathcal{D}_T$. As already observed, $\{T, T_{\mathbf{M}}\} \subset \mathcal{D}_T$. For any t-norm T, some interesting properties of $\mathcal{D}_T$ can be found.

Proposition 5. *Consider a t-norm T and the corresponding class of dominating aggregation operators $\mathcal{D}_T$. Then the following holds:*

(i) For any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{D}_T$, also $\mathbf{D} = \mathbf{A}(\mathbf{B}, \mathbf{C}) \in \mathcal{D}_T$.

(ii) If T is a continuous Archimedean t-norm with an additive generator $f : [0, 1] \to [0, \infty]$, then for any $p, q \in]0, \infty[$, also the weighted t-norm $T_{p,q} \in \mathcal{D}_T$, with $T_{p,q}(x, y) = T\left(x_T^{(p)}, y_T^{(q)}\right)$ and $x_T^{(p)} = f^{(-1)}(p \cdot f(x))$ (see also [13, 9]).

Recall that $\mathcal{D}_T$ was discussed and characterized also in [16, 15] for the case that T is a continuous Archimedean t-norm T with an additive generator $f : [0,1] \to [0,\infty]$.

Proposition 6. *[16] Under the circumstances given above, $\mathbf{A} \in \mathcal{D}_T$ if and only if there is a metric-preserving function $H : [0,\infty]^2 \to [0,\infty]$ such that for all $x, y \in [0,1]$:*

$$f(\mathbf{A}(x,y)) = H(f(x), f(y)).$$

Observe that Proposition 6 is in fact a corollary of Theorem 4. Indeed, H is metric preserving if and only if it is a sub-additive function of two variables, i.e.

$$H(x+y, u+x) \leq H(x,u) + H(y,v)$$

for all $x, y, u, v \in [0,\infty]$, what is, in fact, the domination of the sum operator over H.

3 Special Cases

We will now discuss three special cases of t-norms: $T_{\mathbf{M}} = \min(x,y)$, $T_{\mathbf{P}}(x,y) = x \cdot y$ (by isomorphism any strict t-norm can be covered [13]), $T_{\mathbf{L}}(x,y) = \max(x+y-1,0)$ (by isomorphism, covering all nilpotent t-norms).

Proposition 7. *The class of aggregation operators dominating the minimum t-norm $T_{\mathbf{M}}$ is given by*

$$\mathcal{D}_{\min} = \{\min_{f,g} \mid f, g : [0,1] \to [0,1], \text{ non-decreasing}, \\ f(1) = g(1) = 1, f(0) \cdot g(0) = 0\},$$

where $\min_{f,g} = \min\big(f(x), g(y)\big)$.

Evidently, $\mathbf{A} \in \mathcal{D}_{\min}$ is symmetric if and only if $\mathbf{A}(x,y) = f\big(\min(x,y)\big)$ for some non-decreasing function $f : [0,1] \to [0,1]$ fulfilling $f(0) = 0$ and $f(1) = 1$. Note also, if $\mathbf{A} \in \mathcal{D}_{\min}$, then $\mathbf{A} = \min_{f,g}$, where $f(x) = \mathbf{A}(x,1)$ and $g(y) = \mathbf{A}(1,y)$ (for all $x, y \in [0,1]$).

Concerning $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$, though the classes $\mathcal{D}_{T_{\mathbf{P}}}$ and $\mathcal{D}_{T_{\mathbf{L}}}$ are completely characterized either by Theorem 4 or by Proposition 6, there is no counterpart of Proposition 7 in these cases. However, it is possible to give examples of these members of these classes, and of course, apply Proposition 5 to obtain new members.

Example 8. Observe that $x_{T_{\mathbf{P}}}^{(p)} = x^p$ and thus for all $p, q \in \,]0, \infty[$, the operator $P_{p,q} : [0, 1]^2 \to [0, 1], P_{p,q}(x, y) = x^p y^q$ is contained in $\mathcal{D}_{T_{\mathbf{P}}}$. Particularly, if $p + q = 1$, then $P_{p,q}$ is a weighted geometric mean (compare also examples from [16, 15]).

However, observing that for all $\lambda \geq 1$, the function $H_\lambda : [0, \infty]^2 \to [0, \infty]$, $H_\lambda(x, y) = (x^\lambda + y^\lambda)^{\frac{1}{\lambda}}$, is metric preserving, also any member of the Aczél-Alsina family of t-norms $(T_\lambda^{\mathbf{AA}})_{\lambda \in [1,\infty]}$ (see [13]), is contained in $\mathcal{D}_{T_{\mathbf{P}}}$ because of Proposition 6.

Example 9. Similarly, for all $p, q \in \,]0, \infty[\,, L_{p,q} \in \mathcal{D}_{T_{\mathbf{L}}}$, where $L_{p,q} = T_{\mathbf{L}p,q} = \max(0, px + qy + 1 - p - q)$. In particular, if $p + q = 1$, $L_{p,q}(x, y) = px + qy$, i.e. any weighted mean dominates $T_{\mathbf{L}}$ (compare also examples from [16, 15]).

Based on H_λ any Yager t-norm $T_\lambda^{\mathbf{Y}} \in \mathcal{D}_{T_{\mathbf{L}}}$ whenever $\lambda \geq 1$.

4 Conclusions

An aggregation operator $\mathbf{A}$ preserves T-transitivity of fuzzy relations if and only if it dominates the corresponding t-norm T ($\mathbf{A} \in \mathcal{D}_T$). Although several methods for constructing aggregation operators within a certain class $\mathcal{D}_T$ have been mentioned, an explicit description of $\mathcal{D}_T$ could only been presented for the minimum t-norm $T_{\mathbf{M}}$.

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