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Orderings of Fuzzy Sets Based on Fuzzy Orderings Part II: Generalizations

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Abstract

In Part I of this series of papers, a general approach for ordering fuzzy sets with respect to fuzzy orderings was presented. Part I also highlighted three limitations of this approach. The present paper addresses these limitations and proposes solutions for overcoming them. We first consider a fuzzification of the ordering relation, then ways to compare fuzzy sets with different heights, and finally we introduce how to refine the ordering relation by lexicographic hybridization with a different ordering method.

II.1 Introduction

In the first part of this series of papers [7], a general approach for ordering fuzzy sets with respect to similarity-based fuzzy orderings was presented. The essential features of this approach are its generality and the fact that the restriction to certain subclasses of fuzzy sets is not necessary in advance. The arguments in Section I.4, however, make clear that the most meaningful results are still obtained if fuzzy sets are considered that are normalized and convex (with respect to the crisp or fuzzy ordering considered). The following limitations have been identified:

1. Crisp comparisons of fuzzy sets may be too restrictive (see Subsection I.4.1)
2. Fuzzy sets with different heights are per se incomparable (see Subsection I.4.2)
3. Fuzzy sets with equal (extensional) convex hulls cannot be distinguished (see Subsection I.4.3)

In this paper, we address these three issues, one after the other. The paper is organized as follows. After necessary preliminaries provided in Section II.2, Section II.3 proposes a fuzzification of the ordering relation introduced in the first part. Section II.4 addresses the question how to modify the ordering relation such that fuzzy sets with different heights are not necessarily incomparable anymore. Finally, Section II.5 clarifies how the ordering procedure can be refined by lexicographic composition with a different ordering relation. Lemmata that are necessary for proving some of the results in this paper, but which are not in the core focus of this paper, have been detached and put into an appendix section.

Note that, from now on, we refer to results from Part I without mentioning the reference [7] explicitly. Instead, the prefix "I." indicates that the reference points to an item contained in [7].

II.2 Preliminaries

We adopt all notations and definitions from Section I.2 without any restriction or modification. In this section, we only add concepts that were not yet needed in the first part.

Given a left-continuous t-norm T, the symbol $\vec{T}$ denotes its unique *residual implication* and $\overleftrightarrow{T}$ denotes the corresponding *residual biimplication*:

$$\begin{aligned}
\vec{T}(x,y) &= \sup\{u \in [0,1] \mid T(x,u) \leq y\} \\
\overleftrightarrow{T}(x,y) &= T(\vec{T}(x,y), \vec{T}(y,x)) = \min(\vec{T}(x,y), \vec{T}(y,x))
\end{aligned}$$

We shortly review the most important properties of these two operations (for details, the reader is referred to standard literature [14, 17, 19]).

The residual implication $\vec{T}$ has the following properties (for all $x, y, z \in [0,1]$):

(I1) $x \leq y$ if and only if $\vec{T}(x,y) = 1$

(I2) $T(x,y) \leq z$ if and only if $x \leq \vec{T}(y,z)$

(I3) $T(\vec{T}(x,y), \vec{T}(y,z)) \leq \vec{T}(x,z)$

(I4) $\vec{T}(1,y) = y$

(I5) $T(x, \vec{T}(x,y)) \leq y$

(I6) $T(x,y) \leq \vec{T}(T(x,z), T(y,z))$

Furthermore, $\vec{T}$ is non-increasing and left-continuous in the first argument and non-decreasing and right-continuous in the second argument.

The residual biimplication $\overleftrightarrow{T}$ has the following properties (for all $x, y, z \in [0,1]$):

(B1) $\overleftrightarrow{T}(x,y) = 1$ if and only if $x = y$

(B2) $\overleftrightarrow{T}(x,y) = \overleftrightarrow{T}(y,x)$

(B3) $T(\overleftrightarrow{T}(x,y), \overleftrightarrow{T}(y,z)) \leq \overleftrightarrow{T}(x,z)$

(B4) $\overleftrightarrow{T}(1,y) = y$

(B5) $\overleftrightarrow{T}(x,y) = \vec{T}(\max(x,y), \min(x,y))$

(B6) $\overleftrightarrow{T}(x,y) \geq \min(x,y)$

The *residual negation* induced by the left-continuous t-norm T is defined as follows:

$$N_T(x) = \vec{T}(x,0)$$

For intersecting T-transitive fuzzy relations, the concept of dominance between t-norms is of vital importance [13, 20, 24]. We say that a t-norm T_1 *dominates* another t-norm T_2 if, for every quadruple $x, y, u, v \in [0, 1]$, the following holds:

$$T_1(T_2(x, y), T_2(u, v)) \geq T_2(T_1(x, u), T_1(y, v))$$

As a straightforward extension of the classical crisp inclusion of fuzzy sets, it is easily possible to define a fuzzy inclusion relation of fuzzy sets as

$$\inf_{x \in X} I(A(x), B(x)),$$

where I is some fuzzy implication [2]. Since we are considering a left-continuous t-norm T and since we are interested in meaningful logical properties, we restrict to the following definition:

$$\mathrm{INCL}_T(A, B) = \inf_{x \in X} \vec{T}(A(x), B(x)),$$

In this formula, the infimum can be considered as a generalization of the universal quantifier, as it is standard in many-valued predicate logics based on residuated lattices [10, 18, 19].

It is easy to prove by (I1) that $\mathrm{INCL}_T(A, B) = 1$ if and only if $A \subseteq B$ [4]. Moreover, INCL_T is a fuzzy ordering with respect to T and the following T-equivalence [4]:

$$\mathrm{SIM}_T(A, B) = \inf_{x \in X} \overleftrightarrow{T}(A(x), B(x)).$$

One can also infer easily that $\mathrm{SIM}_T(A, B) = 1$ if and only if $A = B$.

Lemma II.1. *The following holds for all $A, B, C, D \in \mathcal{F}(X)$:*

$$\min(\mathrm{INCL}_T(A, B), \mathrm{INCL}_T(C, D)) \leq \mathrm{INCL}_T(A \cap C, B \cap D) \tag{1}$$

$$\min(\mathrm{SIM}_T(A, B), \mathrm{SIM}_T(C, D)) \leq \mathrm{SIM}_T(A \cap C, B \cap D) \tag{2}$$

Proof. Proofs of inequality (1) are provided in [17, Proposition 2.3] and [18, Proposition 18.2.2]. Further take into account that

$$\mathrm{SIM}_T(A, B) = \min(\mathrm{INCL}_T(A, B), \mathrm{INCL}_T(B, A)) \tag{3}$$

holds for all $A, B \in \mathcal{F}(X)$ [4]. Then the inequality (2) follows by using (1) twice. □

Note that Lemma II.1 is also a corollary of the rather general result [10, Theorem 35].

Lemma II.2. *Consider a binary fuzzy relation $R : X^2 \to [0, 1]$. Then the inequalities*

$$\mathrm{INCL}_T(A, B) \leq \mathrm{INCL}_T(R\uparrow(A), R\uparrow(B)) \tag{4}$$

$$\mathrm{SIM}_T(A, B) \leq \mathrm{SIM}_T(R\uparrow(A), R\uparrow(B)) \tag{5}$$

hold for all $A, B \in \mathcal{F}(X)$, where, for a given fuzzy set $A \in \mathcal{F}(X)$, its image with respect to R *[8, 9, 17] is defined as*

$$R\uparrow(A)(x) = \sup\{T(A(y), R(y, x)) \mid y \in X\}.$$

Proof. Proofs of inequality (4) are provided in [17, Proposition 2.18], [18, Proposition 18.4.1], and [9, Theorem 4.3]. The inequality (5) follows by (3) using (4) twice. □

As corollaries of Lemma II.2, we obtain the following inequalities for a given fuzzy ordering $L : X^2 \to [0,1]$ (as $\mathrm{ATL}(A) = L{\uparrow}(A)$ and $\mathrm{ATM}(A) = L^{-1}{\uparrow}(A)$ holds, where $L^{-1}(x,y) = L(y,x)$ denotes the *inverse fuzzy relation* of L):

$$\mathrm{INCL}_T(A,B) \leq \mathrm{INCL}_T(\mathrm{ATL}(A), \mathrm{ATL}(B)) \tag{6}$$
$$\mathrm{INCL}_T(A,B) \leq \mathrm{INCL}_T(\mathrm{ATM}(A), \mathrm{ATM}(B)) \tag{7}$$
$$\mathrm{SIM}_T(A,B) \leq \mathrm{SIM}_T(\mathrm{ATL}(A), \mathrm{ATL}(B)) \tag{8}$$
$$\mathrm{SIM}_T(A,B) \leq \mathrm{SIM}_T(\mathrm{ATM}(A), \mathrm{ATM}(B)) \tag{9}$$

Recall that a fuzzy set A is called *normalized* if $\mathrm{height}(A) = 1$ and *normal* if there exists an $x \in X$ such that $A(x) = 1$. Additionally, we define

$$\mathrm{kernel}(A) = \{x \in X \mid A(x) = 1\},$$
$$\mathrm{ceiling}(A) = \{x \in X \mid A(x) = \mathrm{height}(A)\}.$$

Obviously, a fuzzy set A is normal if and only if $\mathrm{kernel}(A) \neq \emptyset$. Normality of a fuzzy set A implies that A is normalized, but the converse does not hold in general. We will further use the following three sub-classes of fuzzy sets in this paper:

$$\mathcal{F}_H(X) = \{A \in \mathcal{F}(X) \mid \mathrm{height}(A) = 1\} \qquad \text{(set of normalized fuzzy sets)}$$
$$\mathcal{F}_N(X) = \{A \in \mathcal{F}(X) \mid \mathrm{kernel}(A) \neq \emptyset\} \qquad \text{(set of normal fuzzy sets)}$$
$$\mathcal{F}_T(X) = \{A \in \mathcal{F}(X)\backslash\{\emptyset\} \mid \mathrm{ceiling}(A) \neq \emptyset\}$$

II.3 Fuzzification

In this section, we want to overcome the problem of "artificial preciseness" of a crisp comparison of fuzzy sets as highlighted in Section I.4.1. This will be accomplished by allowing intermediate degrees to which a fuzzy set is smaller than or equal to another. For this purpose, let us recall the definition of $\preceq_L$ (for a given T-E-ordering $L : X^2 \to [0,1]$):

$$A \preceq_L B \;\Leftrightarrow\; \big(\mathrm{ATL}(A) \supseteq \mathrm{ATL}(B) \;\&\; \mathrm{ATM}(A) \subseteq \mathrm{ATM}(B)\big)$$

Given the inclusion relation INCL_T and a t-norm $\tilde{T}$, we can fuzzify $\preceq_L$ in a straightforward way as follows:

$$\mathcal{L}_{\tilde{T},L}(A,B) = \tilde{T}\big(\mathrm{INCL}_T(\mathrm{ATL}(B), \mathrm{ATL}(A)), \mathrm{INCL}_T(\mathrm{ATM}(A), \mathrm{ATM}(B))\big)$$

Let us first clarify in which way the crisp ordering $\preceq_L$ and the fuzzy ordering $\mathcal{L}_L$ are related to each other. The next result gives an exhaustive answer.

Proposition II.3. *Consider a t-norm $\tilde{T}$, a T-equivalence $E : X^2 \to [0,1]$, and a T-E-ordering $L : X^2 \to [0,1]$. Then, for all $A, B \in \mathcal{F}(X)$, $\mathcal{L}_{\tilde{T},L}(A,B) = 1$ holds if and only if $A \preceq_L B$.*

Proof. Immediate consequence of elementary properties of t-norms and the fact that $\mathrm{INCL}_T(A,B)=1$ if and only if $A\subseteq B$. $\square$

The question remains whether $\mathcal{L}_{\tilde{T},L}$ is a fuzzy ordering, and if so, for which T-equivalence.

Theorem II.4. *Consider a t-norm $\tilde{T}$ that dominates T, a T-equivalence $E: X^2\to[0,1]$ and a T-E-ordering $L: X^2\to[0,1]$. Then $\mathcal{L}_{\tilde{T},L}$ is a fuzzy ordering on $\mathcal{F}(X)$ with respect to T and*

$$\mathcal{E}_{\tilde{T},L}(A,B)=\tilde{T}\big(\mathrm{SIM}_T(\mathrm{ATL}(A),\mathrm{ATL}(B)),\mathrm{SIM}_T(\mathrm{ATM}(A),\mathrm{ATM}(B))\big).$$

Moreover, the following holds for all $A,B\in\mathcal{F}(X)$:

$$\mathcal{E}_{\tilde{T},L}(A,B)\leq \mathrm{SIM}_T(\mathrm{ECX}(A),\mathrm{ECX}(B)) \tag{10}$$

For $\tilde{T}=T_{\mathbf{M}}=\min$, even equality is guaranteed to hold (for all $A,B\in\mathcal{F}(X)$):

$$\mathcal{E}_{T_{\mathbf{M}},L}(A,B)= \mathrm{SIM}_T(\mathrm{ECX}(A),\mathrm{ECX}(B)) \tag{11}$$

Proof. Consider the following two fuzzy relations:

$$\begin{aligned}\mathcal{L}_1(A,B)&=\mathrm{INCL}_T(\mathrm{ATL}(B),\mathrm{ATL}(A)),\\ \mathcal{L}_2(A,B)&=\mathrm{INCL}_T(\mathrm{ATM}(A),\mathrm{ATM}(B)).\end{aligned}$$

Both are reflexive and T-transitive, because INCL_T has these properties. We can conclude further, using (3) and [4, Theorem 3.1], that $\mathcal{L}_1$ is a fuzzy ordering with respect to T and

$$\mathcal{E}_1(A,B)=\mathrm{SIM}_T(\mathrm{ATL}(B),\mathrm{ATL}(A))$$

and that $\mathcal{L}_2$ is a fuzzy ordering with respect to T and

$$\mathcal{E}_2(A,B)=\mathrm{SIM}_T(\mathrm{ATM}(A),\mathrm{ATM}(B)).$$

Hence, by [5, Theorem 6.1], we obtain that

$$\mathcal{L}_{\tilde{T},L}(A,B)=\tilde{T}(\mathcal{L}_1(A,B),\mathcal{L}_2(A,B))$$

is a fuzzy ordering with respect to T and

$$\tilde{T}(\mathcal{E}_1(A,B),\mathcal{E}_2(A,B))=\mathcal{E}_{\tilde{T},L}(A,B).$$

The inequality (10) can be proven as follows:

$$\begin{aligned}\mathcal{E}_{\tilde{T},L}(A,B)&=\tilde{T}\big(\mathrm{SIM}_T(\mathrm{ATL}(A),\mathrm{ATL}(B)),\mathrm{SIM}_T(\mathrm{ATM}(A),\mathrm{ATM}(B))\big)\\ &\leq \min\big(\mathrm{SIM}_T(\mathrm{ATL}(A),\mathrm{ATL}(B)),\mathrm{SIM}_T(\mathrm{ATM}(A),\mathrm{ATM}(B))\big)\\ &\overset{(2)}{\leq} \mathrm{SIM}_T(\mathrm{ATL}(A)\cap\mathrm{ATM}(A),\mathrm{ATL}(B)\cap\mathrm{ATM}(B))\\ &=\mathrm{SIM}_T(\mathrm{ECX}(A),\mathrm{ECX}(B))\end{aligned}$$

To complete the proof, consider the case $\tilde{T} = T_{\mathbf{M}} = \min$. Take into account that the equalities $\mathrm{ATL}(\mathrm{ECX}(A)) = \mathrm{ATL}(A)$ and $\mathrm{ATM}(\mathrm{ECX}(A)) = \mathrm{ATM}(A)$ hold (see proof of Theorem I.19). Then we can infer

$$\begin{aligned}\mathrm{SIM}_T(\mathrm{ECX}(A), \mathrm{ECX}(B)) &\overset{(8)}{\leq} \mathrm{SIM}_T(\mathrm{ATL}(\mathrm{ECX}(A)), \mathrm{ATL}(\mathrm{ECX}(B))) \\ &= \mathrm{SIM}_T(\mathrm{ATL}(A), \mathrm{ATL}(B)).\end{aligned}$$

Analogously, we can prove

$$\mathrm{SIM}_T(\mathrm{ECX}(A), \mathrm{ECX}(B)) \leq \mathrm{SIM}_T(\mathrm{ATM}(A), \mathrm{ATM}(B)).$$

Putting these two inequalities together, we obtain

$$\begin{aligned}\mathrm{SIM}_T(\mathrm{ECX}(A), \mathrm{ECX}(B)) &\leq \min\big(\mathrm{SIM}_T(\mathrm{ATL}(A), \mathrm{ATL}(B)), \\ &\qquad\quad \mathrm{SIM}_T(\mathrm{ATM}(A), \mathrm{ATM}(B))\big) \\ &= \mathcal{E}_{T_{\mathbf{M}},L}(A, B).\end{aligned}$$

Together with (10), we finally obtain

$$\mathcal{E}_{T_{\mathbf{M}},L}(A, B) = \mathrm{SIM}_T(\mathrm{ECX}(A), \mathrm{ECX}(B)),$$

and the proof is completed. □

We see that, for $\tilde{T} = T_{\mathbf{M}} = \min$, we have a direct analogy to Theorem I.19. That is why we restrict to this case in the following. To simplify notation, we will drop the index $\tilde{T}$ from now on:

$$\begin{aligned}\mathcal{L}_L(A, B) &= \min\big(\mathrm{INCL}_T(\mathrm{ATL}(B), \mathrm{ATL}(A)), \mathrm{INCL}_T(\mathrm{ATM}(A), \mathrm{ATM}(B))\big) \\ \mathcal{E}_L(A, B) &= \mathrm{SIM}_T(\mathrm{ECX}(A), \mathrm{ECX}(B))\end{aligned}$$

As a simple corollary of Theorem II.4, we obtain that $\mathcal{L}_L$ is fuzzy ordering on $\mathcal{F}(X)$ with respect to T and the T-equivalence $\mathcal{E}_L$.

As $\preceq_L$ is a (crisp) sub-relation of $\mathcal{L}_L$ (see Proposition II.3 above), the comparability of two fuzzy sets with respect to $\mathcal{L}_L$ cannot be worse than the comparability with respect to $\preceq_L$. The following example will show that the problem of artificial strictness when comparing fuzzy sets with $\preceq_L$ is solved very well if we use $\mathcal{L}_L$.

Example II.5. Let us shortly recall the four fuzzy quantities A_1, B_1, A_2 and B_2 from Example I.16. We obtain

$$\begin{aligned}\mathcal{L}_L(A_1, B_1) &= 1, & \mathcal{L}_L(B_1, A_1) &= 0, \\ \mathcal{L}_L(A_2, B_2) &= 1, & \mathcal{L}_L(B_2, A_2) &= 0,\end{aligned}$$

if we define L to be the crisp linear ordering of real numbers (in these two cases, the results are even independent of the choice of the t-norm T).

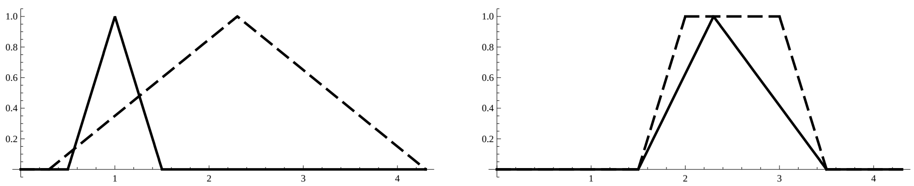

Figure 1: Left: two convex fuzzy quantities A_3 (solid line) and B_3 (dashed line); right: two convex fuzzy quantities A_4 (solid line) and B_4 (dashed line).

Now we reconsider the fuzzy quantities A_3 and B_3 from Example I.17 (for convenience, depicted in Figure 1 on the left side again). Let us use $T = T_{\mathbf{L}}$ and define L to be the crisp linear ordering of real numbers again. It is easy to see that

$$\begin{aligned}\mathrm{INCL}_{T_{\mathbf{L}}}(A, B) &= \inf_{x \in X} \vec{T}_{\mathbf{L}}(A(x), B(x)) \\ &= \inf_{x \in X} \min(1, 1 - A(x) + B(x)) \\ &= 1 - \sup\{A(x) - B(x) \mid x \in X \,\&\, A(x) > B(x)\}\end{aligned}$$

holds, i.e. $\mathrm{INCL}_{T_{\mathbf{L}}}(A, B)$ is one minus the maximal degree to which A exceeds B. Taken this into account, one can easily compute

$$\mathcal{L}_L(A_3, B_3) = 0.9, \qquad\qquad \mathcal{L}_L(B_3, A_3) = 0,$$

which appears to be a reasonable result.

On the right-hand side, Figure 1 shows two fuzzy quantities A_4 and B_4. For these two fuzzy sets, we get the following (using $T = T_{\mathbf{L}}$ and the crisp linear ordering of real numbers again):

$$\mathcal{L}_L(A_4, B_4) = \frac{5}{8} = 0.625 \qquad\qquad \mathcal{L}_L(B_4, A_4) = \frac{5}{12} = 0.41\dot{6}$$

Note that A_4 and B_4 are incomparable with respect to $\preceq_L$. We see that, even in this peculiar case which seems hopeless for extension principle-based approaches, quite a sensitive result is obtained.

II.4 Comparing Fuzzy Sets With Different Heights

As pointed out in Subsection I.4.2, equal heights of two fuzzy sets are a necessary condition for the comparability with respect to $\preceq_L$. The height of a fuzzy set—if we adopt the viewpoint of fuzzy predicate logics [10, 18, 19]—can be interpreted as the truth degree to which there exists an element contained in this fuzzy set. One can argue that two fuzzy sets need not be comparable if these degrees do not coincide. On the other hand, if we view fuzzy sets more pragmatically as $X \to [0, 1]$ functions, it is not immediate that different heights, in particular, if they are almost the same, should make such a big difference.

Let us first investigate by means of two simple examples whether the fuzzification proposed in Section II.3 already constitutes a (partial) solution to this issue.

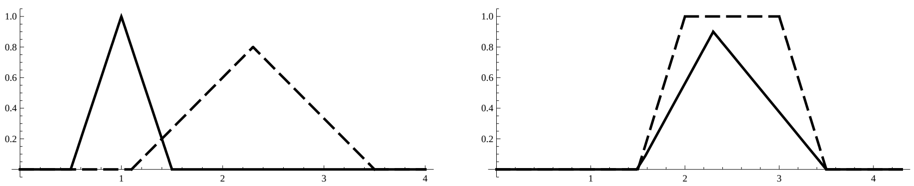

Figure 2: Left: two convex fuzzy quantities A_5 (solid line) and B_5 (dashed line); right: two convex fuzzy quantities A_5 (solid line) and B_6 (dashed line).

Example II.6. Consider the four fuzzy quantities A_5, B_5, A_6 and B_6 shown in Figure 2. It is obvious that A_5 and B_5 are incomparable with respect to $\preceq_L$, no matter which crisp or fuzzy ordering L we use (compare with Proposition I.23). The same is true for A_6 and B_6. If we use $T = T_{\mathbf{L}}$ and if we define L to be the crisp linear ordering of real numbers, we obtain the following:

$$\begin{aligned} \mathcal{L}_L(A_5, B_5) &= 0.8 & \qquad \mathcal{L}_L(B_5, A_5) &= 0 \\ \mathcal{L}_L(A_6, B_6) &= 0.5625 & \qquad \mathcal{L}_L(B_6, A_6) &= 0.375 \end{aligned}$$

So, we obtain results that appear intuitively reasonable.

The question remains whether we could still use some tricks to enforce comparability of fuzzy sets with different heights.

The simplest idea is *linear normalization*, i.e. to re-scale the membership functions such that the height of the resulting fuzzy set is 1. More specifically, for a fuzzy set $A \in \mathcal{F}(X)\backslash\{\emptyset\}$, we can define

$$\bar{A}(x) = \frac{A(x)}{\mathrm{height}(A)}.$$

It is clear that $\mathrm{height}(\bar{A}) = 1$ holds then. We have to point out, however, that two fuzzy sets A and B with different heights are scaled differently then, which can lead to distorted results. Therefore, we do not want to pursue this approach.

B. Moser and L. T. Kóczy [21,23] have brought up the idea to use α-cuts somehow, but only up to the minimum of the heights of the two fuzzy sets considered. Returning to α-cuts would mean a step back, because they are unhandy. This issue could be solved quite easily (see [3, Subsection 6.3.2] for details), but the problem arises that the resulting ordering relation is not transitive anymore. To see that, consider the three fuzzy quantities C_1, C_2 and C_3 shown in Figure 3. Obviously, we have $\mathrm{height}(C_1) = 0.8$, $\mathrm{height}(C_2) = 0.5$ and $\mathrm{height}(C_3) = 1$. Let us compare these three fuzzy sets with $\preceq_I$, but only up to the largest common level (i.e. up to the minimum of the two heights),[1] and let us call this relation $\preceq'_I$. Then we obtain that $C_1 \preceq'_I C_2$ (the largest common level of C_1 and C_2 is 0.5) and $C_2 \preceq'_I C_3$ (the largest common level of C_2 and C_3 is 0.5, too). However, we see that $C_1 \preceq'_I C_3$ does *not* hold: the largest common level of C_1 and C_3 is 0.8 and the two peaks are *not* in proper order.

[1] The ordering relation $\preceq_I$ can safely be applied to α-cuts if we consider Proposition I.13.

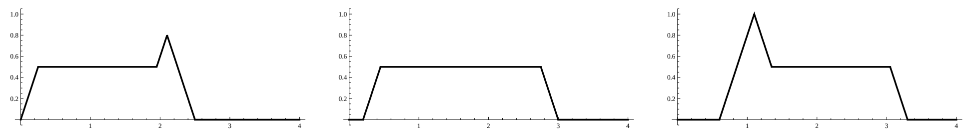

Figure 3: Three fuzzy quantities C_1, C_2 and C_3 (from left to right).

Therefore, let us propose a different approach. From an intuitive point of view, the positions of the highest truth values play a fundamental role when ordering fuzzy sets. We use these positions to enforce a height of 1 without distorting the other truth degrees.

Definition II.7. For all $A \in \mathcal{F}(X)$, we define the fuzzy set $\lceil A \rceil$, the *lifting* of A, as follows:

$$\lceil A \rceil(x) = \begin{cases} 1 & \text{if } A(x) = \mathrm{height}(A), \\ A(x) & \text{otherwise.} \end{cases}$$

As $A(x) = \mathrm{height}(A)$ if and only if $x \in \mathrm{ceiling}(A)$, we can reformulate the previous definition as follows (if we adopt the usual viewpoint that every crisp set, by its characteristic function, is also a fuzzy set):

$$\lceil A \rceil = A \cup \mathrm{ceiling}(A) \tag{12}$$

The next lemma shows that, under some mild conditions, lifting the ceiling ensures that the resulting fuzzy set is normalized.

Lemma II.8. *Provided that $A \in \mathcal{F}_H(X) \cup \mathcal{F}_T(X)$, the equality* $\mathrm{height}(\lceil A \rceil) = 1$ *holds, i.e. $\lceil A \rceil \in \mathcal{F}_H(X)$. If $A \in \mathcal{F}_T(X)$, then $\lceil A \rceil$ is normal, i.e. $\lceil A \rceil \in \mathcal{F}_N(X)$.*

Proof. If $A \in \mathcal{F}_H(X)$, i.e. $\mathrm{height}(A) = 1$, the equality $A = \lceil A \rceil$ holds trivially, and so does $\mathrm{height}(\lceil A \rceil) = \mathrm{height}(A) = 1$. If, on the other hand, $\mathrm{ceiling}(A) \neq \emptyset$, then $\lceil A \rceil$ is normal. □

The following lemma provides us with some basic properties of lifting.

Lemma II.9. *Consider two fuzzy sets $A, B \in \mathcal{F}(X)$.*

1. *$\lceil A \rceil = \lceil B \rceil$ holds if and only if* $\mathrm{ceiling}(A) = \mathrm{ceiling}(B)$ *and $A(x) = B(x)$ for all $x \notin \mathrm{ceiling}(A)$.*
2. *If* $\mathrm{height}(A) = \mathrm{height}(B)$ *holds, the following representation holds:*

$$\lceil A \cap B \rceil = \lceil A \rceil \cap \lceil B \rceil \tag{13}$$

Proof. To prove 1., assume that $\lceil A \rceil = \lceil B \rceil$ holds. If $\lceil A \rceil(x) = \lceil B \rceil(x) = 1$ holds, then $x \in \mathrm{ceiling}(A)$ and $x \in \mathrm{ceiling}(B)$. If, however, $\lceil A \rceil(x) = \lceil B \rceil(x) < 1$ holds, it is clear that $x \notin \mathrm{ceiling}(A)$, $x \notin \mathrm{ceiling}(B)$ and $A(x) = B(x)$. The converse implication is trivial.

Now assume $\text{height}(A) = \text{height}(B)$. To prove the equality (13), we have to prove

$$\lceil A \cap B \rceil(x) = \min(\lceil A \rceil(x), \lceil B \rceil(x)) \tag{14}$$

for every $x \in X$. So let us consider an arbitrary $x \in X$ and distinguish the following four cases:

1. $x \in \text{ceiling}(A)$ and $x \in \text{ceiling}(B)$: in this case, we have

   $$(A \cap B)(x) = \min(A(x), B(x)) = \min(\text{height}(A), \text{height}(B)) = \text{height}(A),$$

   i.e. $x \in \text{ceiling}(A \cap B)$, and we have 1 on both sides of (14);

2. $x \in \text{ceiling}(A)$ and $x \notin \text{ceiling}(B)$: in this case, we have

   $$(A \cap B)(x) = \min(A(x), B(x)) = \min(\text{height}(A), B(x)) = B(x)$$

   on the left-hand side and obviously the same on the right-hand side of (14);

3. $x \notin \text{ceiling}(A)$ and $x \in \text{ceiling}(B)$: analogously, to the previous case, we can infer that we have $A(x)$ on both sides of the equality (14);

4. $x \notin \text{ceiling}(A)$ and $x \notin \text{ceiling}(B)$: here we obtain

   $$(A \cap B)(x) = \min(A(x), B(x)) < \text{height}(A),$$

   which implies that we have $\min(A(x), B(x))$ on both sides of the equation (14). □

Now we define a modified relation $\preceq'_L$ that overcomes the problem of incomparability caused by different heights. This is done by comparing the liftings of left and right flanks of the two fuzzy sets considered.[2]

Definition II.10. Consider a T-equivalence $E : X^2 \to [0, 1]$ and a T-E-ordering $L : X^2 \to [0, 1]$. Then the relation $\preceq'_L$ is defined in the following way:

$$A \preceq'_L B \;\Leftrightarrow\; \big(\lceil \text{ATL}(A) \rceil \supseteq \lceil \text{ATL}(B) \rceil \;\&\; \lceil \text{ATM}(A) \rceil \subseteq \lceil \text{ATM}(B) \rceil\big)$$

It is obvious that the problem of incomparability due to different heights reduces to cases in which the ceiling of $\text{ATL}(A)$, $\text{ATM}(A)$, $\text{ATL}(B)$ or $\text{ATM}(B)$ is empty. It is clear that $\text{ceiling}(A) \neq \emptyset$ is a sufficient condition for $\text{ceiling}(\text{ATL}(A)) \neq \emptyset$ and $\text{ceiling}(\text{ATM}(A)) \neq \emptyset$. If X is a finite set, this problem cannot occur at all, since $\text{ceiling}(A) \neq \emptyset$ holds trivially for all fuzzy subsets of finite domains. In the case that $X \subseteq \mathbb{R}$ and that L fuzzifies the crisp linear ordering of real numbers, it is possible to prove that the compactness of X is sufficient to ensure $\text{ceiling}(\text{ATL}(A)) \neq \emptyset$ and $\text{ceiling}(\text{ATM}(A)) \neq \emptyset$. We omit such difficile considerations in the following and restrict to fuzzy sets from $\mathcal{F}_H(X) \cup \mathcal{F}_T(X)$ for simplicity.

It is clearly desirable that $A \preceq'_L B$ holds if $A \preceq_L B$ holds, i.e. that $\preceq'_L$ is an extension of $\preceq_L$. The following result proves this property.

[2] An earlier version of this work [3, Subsection 6.3.2] used a simpler definition $A \preceq'_L B \Leftrightarrow \lceil A \rceil \preceq_L \lceil B \rceil$, but this approach turned out to have severe shortcomings.

Proposition II.11. *Consider a T-equivalence $E : X^2 \to [0,1]$, a T-E-ordering $L : X^2 \to [0,1]$, and two fuzzy sets $A, B \in \mathcal{F}_H(X) \cup \mathcal{F}_T(X)$. Then $A \preceq_L B$ implies $A \preceq'_L B$.*

Proof. We know that $A \preceq_L B$ implies $\mathrm{height}(A) = \mathrm{height}(B)$ (cf. Proposition I.23). So, if both fuzzy sets are normalized (i.e. from $\mathcal{F}_H(X)$), nothing is to prove, as $A \preceq'_L B$ if and only if $A \preceq_L B$. So assume that $A, B \in \mathcal{F}_T(X)$ holds (regardless of whether the heights are 1 or lower). Then we can be sure that $\mathrm{ATL}(A) \in \mathcal{F}_T(X)$, $\mathrm{ATM}(A) \in \mathcal{F}_T(X)$, $\mathrm{ATL}(B) \in \mathcal{F}_T(X)$ and $\mathrm{ATM}(B) \in \mathcal{F}_T(X)$.

Now consider an $x \in \mathrm{ceiling}(\mathrm{ATL}(B))$. Then the following chain of inequalities follows from $\mathrm{ATL}(A) \supseteq \mathrm{ATL}(B)$ (using also Lemma I.22):

$$\begin{aligned}\mathrm{height}(A) = \mathrm{height}(\mathrm{ATL}(A)) &\geq \mathrm{ATL}(A)(x)\\ &\geq \mathrm{ATL}(B)(x) = \mathrm{height}(\mathrm{ATL}(B)) = \mathrm{height}(B)\end{aligned}$$

Hence, using $\mathrm{height}(A) = \mathrm{height}(B)$, we know that $x \in \mathrm{ceiling}(\mathrm{ATL}(A))$ holds, which finally implies

$$\mathrm{ceiling}(\mathrm{ATL}(A)) \supseteq \mathrm{ceiling}(\mathrm{ATL}(B)).$$

From (12), therefore, we get

$$\begin{aligned}\lceil \mathrm{ATL}(A) \rceil &= \mathrm{ATL}(A) \cup \mathrm{ceiling}(\mathrm{ATL}(A))\\ &\supseteq \mathrm{ATL}(B) \cup \mathrm{ceiling}(\mathrm{ATL}(B)) = \lceil \mathrm{ATL}(B) \rceil.\end{aligned}$$

Analogously, we can prove $\lceil \mathrm{ATM}(A) \rceil \subseteq \lceil \mathrm{ATM}(B) \rceil$, and the assertion we had to prove follows directly. □

Before we turn back to investigating the properties of the relation $\preceq'_L$, we prove a helpful lemma for representing the lifting of the extensional convex hull of a fuzzy set.

Lemma II.12. *Consider a T-equivalence $E : X^2 \to [0,1]$ and a T-E-ordering $L : X^2 \to [0,1]$. Then the following holds for all $A \in \mathcal{F}(X)$:*

$$\lceil \mathrm{ECX}(A) \rceil = \lceil \mathrm{ATL}(A) \rceil \cap \lceil \mathrm{ATM}(B) \rceil$$

Proof. We know from Lemma I.22 that $\mathrm{height}(\mathrm{ATL}(A)) = \mathrm{height}(\mathrm{ATM}(A))$. Using the definition $\mathrm{ECX}(A) = \mathrm{ATL}(A) \cap \mathrm{ATM}(A)$ and Lemma II.9, 2., the assertion follows. □

The following theorem provides us with a characterization of the symmetric kernel of $\preceq'_L$, i.e. with a characterization in which cases $\preceq'_L$ violates antisymmetry. We see that two fuzzy sets can only be indistinguishable with respect to $\preceq'_L$ if the liftings of their extensional convex hulls coincide. For proving the converse implication, we need to assume that L is strongly complete.

Theorem II.13. *Consider a T-equivalence $E : X^2 \to [0,1]$ and a T-E-ordering $L : X^2 \to [0,1]$. Then the relation $\preceq'_L$ is a preordering on $\mathcal{F}_H(X) \cup \mathcal{F}_T(X)$. If we denote its symmetric kernel with $\cong'_L$, the following implication holds for all $A, B \in \mathcal{F}_H(X) \cup \mathcal{F}_T(X)$:*

$$A \cong'_L B \;\Rightarrow\; \lceil \mathrm{ECX}(A) \rceil = \lceil \mathrm{ECX}(B) \rceil$$

If L is strongly complete, even the converse implication holds, and $\cong'_L$ can be characterized as follows (for all $A, B \in \mathcal{F}_H(X) \cup \mathcal{F}_T(X)$):

$$A \cong'_L B \;\Leftrightarrow\; \lceil \mathrm{ECX}(A) \rceil = \lceil \mathrm{ECX}(B) \rceil$$

Proof. That $\preceq'_L$ is a preordering follows trivially from the fact that the inclusion relation $\subseteq$ is reflexive and transitive.

Now assume that, for two fuzzy sets $A, B \in \mathcal{F}_H(X) \cup \mathcal{F}_T(X)$, $A \cong'_L B$ holds, which is nothing else but $\lceil \mathrm{ATL}(A) \rceil = \lceil \mathrm{ATL}(B) \rceil$ and $\lceil \mathrm{ATM}(A) \rceil = \lceil \mathrm{ATM}(B) \rceil$. Then $\lceil \mathrm{ECX}(A) \rceil = \lceil \mathrm{ECX}(B) \rceil$ follows from Lemma II.12.

Conversely, assume that that L is strongly complete and that $\lceil \mathrm{ECX}(A) \rceil = \lceil \mathrm{ECX}(B) \rceil$ holds. Lemma II.22 ensures the existence of two three-set partitions (X_l^A, X_m^A, X_r^A) and (X_l^B, X_m^B, X_r^B) such that the representations of this lemma are valid for A and B, respectively. From Lemma II.9, 1., we can infer $\mathrm{ceiling}(\mathrm{ECX}(A)) = \mathrm{ceiling}(\mathrm{ECX}(B))$ which is nothing else but the equality $X_m^A = X_m^B$. Lemma II.22, moreover, implies that

$$\begin{aligned} \mathrm{ATL}(A)(x) &= \mathrm{ATM}(A)(x) = \mathrm{height}(A) \\ \mathrm{ATL}(B)(x) &= \mathrm{ATM}(B)(x) = \mathrm{height}(B) \end{aligned}$$

holds for all $x \in X_m^A = X_m^B$. Now let us choose an $x \notin X_m^A = X_m^B$ (which already implies $\mathrm{ECX}(A)(x) = \mathrm{ECX}(B)(x)$ by Lemma II.9, 1.). Since (X_l^A, X_m^A, X_r^A) is a partition, either $x \in X_l^A$ or $x \in X_r^A$ holds and either $x \in X_l^B$ or $x \in X_r^B$ must hold. From $X_m^A = X_m^B$ and Lemma II.22, 1., it follows that $x \in X_l^A$ holds if and only if $x \in X_l^B$ and $x \in X_r^A$ holds if and only if $x \in X_r^B$. Thus, we have $X_l^A = X_l^B$ and $X_r^A = X_r^B$. As a first case, assume that $x \in X_l^A = X_l^B$. Then we have

$$\mathrm{ATL}(A)(x) = \mathrm{ECX}(A)(x) = \mathrm{ECX}(B)(x) = \mathrm{ATL}(B)(x),$$

whereas $\mathrm{ATM}(A)(x) = \mathrm{height}(A)$ and $\mathrm{ATM}(B)(x) = \mathrm{height}(B)$ holds. Analogously, we can infer

$$\mathrm{ATM}(A)(x) = \mathrm{ECX}(A)(x) = \mathrm{ECX}(B)(x) = \mathrm{ATM}(B)(x),$$

whereas $\mathrm{ATL}(A)(x) = \mathrm{height}(A)$ and $\mathrm{ATL}(B)(x) = \mathrm{height}(B)$ for the case that $x \in X_r^A = X_r^B$. We can summarize that $\mathrm{ATL}(A)(x) = \mathrm{ATL}(B)(x)$ for all $x \in X_l^A = X_l^B$, while $\mathrm{ATL}(A)(x) = \mathrm{height}(A)$ and $\mathrm{ATL}(B)(x) = \mathrm{height}(B)$ for all $x \notin X_l^A$. Thus, we can infer that $\lceil \mathrm{ATL}(A) \rceil = \lceil \mathrm{ATL}(B) \rceil$. Analogously, we obtain the equality $\lceil \mathrm{ATM}(A) \rceil = \lceil \mathrm{ATM}(B) \rceil$, which completes the proof. $\square$

The following example demonstrates that the requirement of strong completeness in the last assertion of Theorem II.13 cannot be omitted.

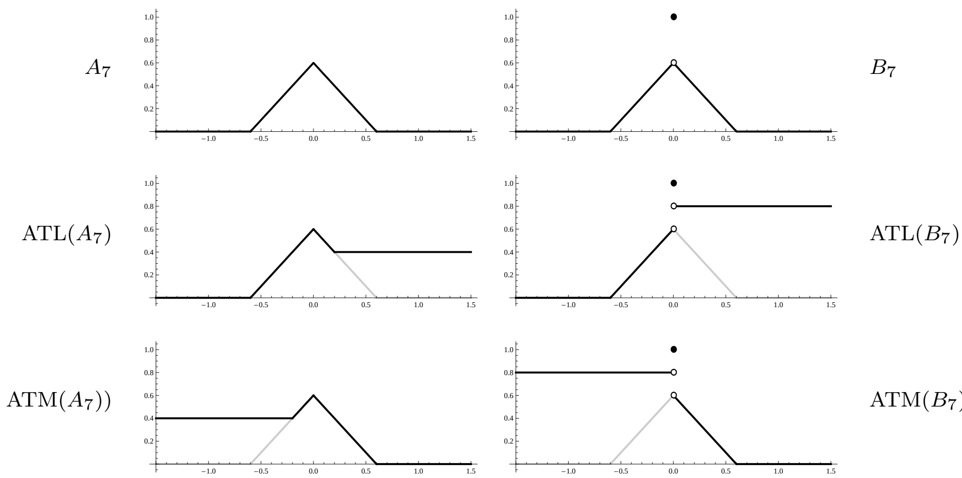

Figure 4: The two fuzzy quantities A_7 and B_7 and results that are obtained when the two operators ATL and ATM are applied to them.

Example II.14. Consider $X = \mathbb{R}$, the Łukasiewicz t-norm $T_{\mathbf{L}}$, and the following fuzzy relation:

$$L(x, y) = \begin{cases} 0 & \text{if } x > y \\ 1 & \text{if } x = y \\ 0.8 & \text{if } x < y \end{cases}$$

It is easy to see that L is a fuzzy ordering with respect to $T_{\mathbf{L}}$ and the crisp equality of real numbers. Obviously, L is not strongly complete. Now consider the following two fuzzy quantities:

$$A_7(x) = \max(0, 0.6 - |x|) \qquad\qquad B_7(x) = \begin{cases} 1 & \text{if } x = 0 \\ A_7(x) & \text{otherwise} \end{cases}$$

Obviously, $\mathrm{ceiling}(A_7) = \mathrm{ceiling}(B_7) = \{0\}$ and $A_7 \neq \lceil A_7 \rceil = \lceil B_7 \rceil = B_7$ hold. The plots in Figure 4 illustrate that $\lceil \mathrm{ATL}(A_7) \rceil \neq \lceil \mathrm{ATL}(B_7) \rceil$ and $\lceil \mathrm{ATM}(A_7) \rceil \neq \lceil \mathrm{ATM}(B_7) \rceil$ hold, while obviously $\lceil \mathrm{ECX}(A_7) \rceil = \lceil \mathrm{ECX}(B_7) \rceil = B_7$ holds. So we have constructed examples of two fuzzy quantities A_7 and B_7 whose lifted extensional convex hulls coincide, while $A_7 \cong'_L B_7$ does not hold.

The seemingly weaker result for fuzzy orderings that are not strongly complete is not too big an eyesore. It is desirable to have a symmetric kernel that is as small as possible; the first result in Theorem II.13 ensures that $\cong'_L$ cannot exceed the equivalence relation defined as $\lceil \mathrm{ECX}(A) \rceil = \lceil \mathrm{ECX}(B) \rceil$, and in the case that L is strongly complete, the two equivalence relations coincide.

Let us shortly reconsider the examples that we have dealt with so far to see how the results change if we apply $\preceq'_L$.

Example II.15. Examples I.16 and I.17 only dealt with normal fuzzy quantities. For these examples, the results do not change at all if we replace $\preceq_L$ by $\preceq'_L$, since $\preceq'_L$ coincides with $\preceq_L$ as long as fuzzy sets with the same height are considered (cf. Proposition II.11). The same is true for the fuzzy quantities A_4 and B_4 introduced in Example II.5.

Now let us consider the fuzzy quantities of Example II.6. We obtain that $A_5 \preceq'_L B_5$ and $B_5 \not\preceq'_L A_5$—a meaningful result that is perfectly in accordance with the initial motivation behind $\preceq'_L$. It is further easy to see that A_6 and B_6 are incomparable with respect to $\preceq'_L$, i.e. neither $A_6 \preceq'_L B_6$ nor $B_6 \preceq'_L A_6$ holds. The three fuzzy quantities C_1, C_2 and C_3 (see Figure 3) are mutually incomparable with respect to $\preceq'_L$.

Fuzzification can be carried out analogously to Section II.3. So let us define a fuzzy relation on $\mathcal{F}(X)$ in the following way:

$$\begin{aligned}\mathcal{L}'_L(A,B) = \min\big(&\mathrm{INCL}_T(\lceil \mathrm{ATL}(B)\rceil, \lceil \mathrm{ATL}(A)\rceil),\\ &\mathrm{INCL}_T(\lceil \mathrm{ATM}(B)\rceil, \lceil \mathrm{ATM}(A)\rceil)\big)\end{aligned}$$

It is immediate (analogously to Proposition II.3) that $\mathcal{L}'_L(A,B) = 1$ if and only if $A \preceq'_L B$ holds.

The next theorem clarifies the properties of the fuzzy relation $\mathcal{L}'_L$. It can be understood both as an analogue of Theorem II.4 and a fuzzification of Theorem II.13.

Theorem II.16. *Consider a T-equivalence $E : X^2 \to [0,1]$ and a T-E-ordering $L : X^2 \to [0,1]$. Then the fuzzy relation $\mathcal{L}'_L$ is a T-$\mathcal{E}'_L$-ordering on $\mathcal{F}(X)$, where*

$$\mathcal{E}'_L(A,B) = \min\big(\mathrm{SIM}_T(\lceil \mathrm{ATL}(A)\rceil, \lceil \mathrm{ATL}(B)\rceil), \mathrm{SIM}_T(\lceil \mathrm{ATM}(A)\rceil, \lceil \mathrm{ATM}(B)\rceil)\big).$$

Furthermore, the inequality

$$\mathcal{E}'_L(A,B) \leq \mathrm{SIM}_T(\lceil \mathrm{ECX}(A)\rceil, \lceil \mathrm{ECX}(B)\rceil) \tag{15}$$

holds for all $A, B \in \mathcal{F}_T(X) \cup \mathcal{F}_H(X)$. Provided that L is strongly complete,

$$\mathcal{E}'_L(A,B) = \mathrm{SIM}_T(\lceil \mathrm{ECX}(A)\rceil, \lceil \mathrm{ECX}(B)\rceil) \tag{16}$$

holds for all $A, B \in \mathcal{F}_T(X) \cup \mathcal{F}_H(X)$.

Proof. Analogously to the proof of Theorem II.4 (see also [5, Theorem 6.1]), we can infer the following:

1. $\mathcal{L}'_L$ is reflexive and T-transitive, i.e. a T-preordering on $\mathcal{F}(X)$;
2. The symmetric kernel of $\mathcal{L}'_L$ with respect to the minimum,
$$\mathcal{E}'_L(A,B) = \min(\mathcal{L}'_L(A,B), \mathcal{L}'_L(B,A)),$$
is a T-equivalence on $\mathcal{F}(X)$;
3. $\mathcal{L}'_L$ is a T-$\mathcal{E}'_L$-ordering.

Then the representation of $\mathcal{E}'_L$ follows from (3) in the following way:

$$\begin{aligned}
\mathcal{E}'_L(A,B) &= \min(\mathcal{L}'_L(A,B), \mathcal{L}'_L(B,A)) \\
&= \min(\mathrm{INCL}_T(\lceil \mathrm{ATL}(B)\rceil, \lceil \mathrm{ATL}(A)\rceil), \mathrm{INCL}_T(\lceil \mathrm{ATM}(B)\rceil, \lceil \mathrm{ATM}(A)\rceil), \\
&\qquad \mathrm{INCL}_T(\lceil \mathrm{ATL}(A)\rceil, \lceil \mathrm{ATL}(B)\rceil), \mathrm{INCL}_T(\lceil \mathrm{ATM}(A)\rceil, \lceil \mathrm{ATM}(B)\rceil)) \\
&= \min\big(\mathrm{SIM}_T(\lceil \mathrm{ATL}(A)\rceil, \lceil \mathrm{ATL}(B)\rceil), \mathrm{SIM}_T(\lceil \mathrm{ATM}(A)\rceil, \lceil \mathrm{ATM}(B)\rceil)\big)
\end{aligned}$$

The inequality (15) then follows from Lemma II.1, (2), and Lemma II.12:

$$\begin{aligned}
\mathcal{E}'_L(A,B) &= \min\big(\mathrm{SIM}_T(\lceil \mathrm{ATL}(A)\rceil, \lceil \mathrm{ATL}(B)\rceil), \mathrm{SIM}_T(\lceil \mathrm{ATM}(A)\rceil, \lceil \mathrm{ATM}(B)\rceil)\big) \\
&\leq \mathrm{SIM}_T(\lceil \mathrm{ATL}(A)\rceil \cap \lceil \mathrm{ATM}(A)\rceil, \lceil \mathrm{ATL}(B)\rceil \cap \lceil \mathrm{ATM}(B)\rceil) \\
&= \mathrm{SIM}_T(\lceil \mathrm{ECX}(A)\rceil, \lceil \mathrm{ECX}(B)\rceil)
\end{aligned}$$

In order to prove the equality (16), consider the following:

$$\begin{aligned}
\mathcal{E}'_L(A,B) &= \min\big(\mathrm{SIM}_T(\lceil \mathrm{ATL}(A)\rceil, \lceil \mathrm{ATL}(B)\rceil), \mathrm{SIM}_T(\lceil \mathrm{ATM}(A)\rceil, \lceil \mathrm{ATM}(B)\rceil)\big) \\
&= \min\big(\inf_{x\in X} \overleftrightarrow{T}(\lceil \mathrm{ATL}(A)\rceil(x), \lceil \mathrm{ATL}(B)\rceil(x)), \\
&\qquad \inf_{x\in X} \overleftrightarrow{T}(\lceil \mathrm{ATM}(A)\rceil(x), \lceil \mathrm{ATM}(B)\rceil(x))\big) \\
&= \inf_{x\in X} \min\big(\overleftrightarrow{T}(\lceil \mathrm{ATL}(A)\rceil(x), \lceil \mathrm{ATL}(B)\rceil(x)), \\
&\qquad \overleftrightarrow{T}(\lceil \mathrm{ATM}(A)\rceil(x), \lceil \mathrm{ATM}(B)\rceil(x))\big)
\end{aligned}$$

Let us define the following two functions (for all $x \in X$):

$$\begin{aligned}
f_1(x) &= \min\big(\overleftrightarrow{T}(\lceil \mathrm{ATL}(A)\rceil(x), \lceil \mathrm{ATL}(B)\rceil(x)), \overleftrightarrow{T}(\lceil \mathrm{ATM}(A)\rceil(x), \lceil \mathrm{ATM}(B)\rceil(x))\big) \\
f_2(x) &= \overleftrightarrow{T}(\lceil \mathrm{ECX}(A)\rceil(x), \lceil \mathrm{ECX}(B)\rceil(x))
\end{aligned}$$

Then (16) is obviously equivalent to the equality

$$\inf_{x\in X} f_1(x) = \inf_{x\in X} f_2(x) \tag{17}$$

Assuming that L is strongly complete, Lemma II.22 ensures that we can define two three-set partitions (X_l^A, X_m^A, X_r^A) and (X_l^B, X_m^B, X_r^B) such that the equalities listed in this lemma are valid for A and B, respectively. Let us define the following nine subsets of X:

$$\begin{array}{lll}
X_1 = X_l^A \cap X_l^B & X_2 = X_l^A \cap X_m^B & X_3 = X_l^A \cap X_r^B \\
X_4 = X_m^A \cap X_l^B & X_5 = X_m^A \cap X_m^B & X_6 = X_m^A \cap X_r^B \\
X_7 = X_r^A \cap X_l^B & X_8 = X_r^A \cap X_m^B & X_9 = X_r^A \cap X_r^B
\end{array}$$

The family $(X_1, \ldots, X_9)$ is also a partition of X, since $(X_1, \ldots, X_9)$ is the joint refinement of (X_l^A, X_m^A, X_r^A) and (X_l^B, X_m^B, X_r^B). For these nine sets, we can infer simplified representations of $f_1(x)$ and $f_2(x)$ as follows (making use of basic properties of $\overleftrightarrow{T}$ and the representations provided by Lemma II.22):

1. $x \in X_1 = X_l^A \cap X_l^B$:

$$\begin{aligned} f_1(x) &= \min\big(\vec{T}(\mathrm{ATL}(A)(x), \mathrm{ATL}(B)(x)), \vec{T}(1,1)\big) \\ &= \vec{T}(\mathrm{ATL}(A)(x), \mathrm{ATL}(B)(x)) \\ f_2(x) &= \vec{T}(\mathrm{ATL}(A)(x), \mathrm{ATL}(B)(x)) \end{aligned}$$

2. $x \in X_2 = X_l^A \cap X_m^B$:

$$\begin{aligned} f_1(x) &= \min\big(\vec{T}(\mathrm{ATL}(A)(x), 1), \vec{T}(1,1)\big) = \mathrm{ATL}(A)(x) \\ f_2(x) &= \vec{T}(\mathrm{ATL}(A)(x), 1) = \mathrm{ATL}(A)(x) \end{aligned}$$

3. $x \in X_3 = X_l^A \cap X_r^B$:

$$\begin{aligned} f_1(x) &= \min\big(\vec{T}(\mathrm{ATL}(A)(x), 1), \vec{T}(1, \mathrm{ATM}(B)(x))\big) \\ &= \min(\mathrm{ATL}(A)(x), \mathrm{ATM}(B)(x)) \\ f_2(x) &= \vec{T}(\mathrm{ATL}(A)(x), \mathrm{ATM}(B)(x)) \end{aligned}$$

4. $x \in X_4 = X_m^A \cap X_l^B$:

$$\begin{aligned} f_1(x) &= \min\big(\vec{T}(1, \mathrm{ATL}(B)(x)), \vec{T}(1,1)\big) = \mathrm{ATL}(B)(x) \\ f_2(x) &= \vec{T}(1, \mathrm{ATL}(B)(x)) = \mathrm{ATL}(B)(x) \end{aligned}$$

5. $x \in X_5 = X_m^A \cap X_m^B$:

$$\begin{aligned} f_1(x) &= \min\big(\vec{T}(1,1), \vec{T}(1,1)\big) = 1 \\ f_2(x) &= \vec{T}(1,1) = 1 \end{aligned}$$

6. $x \in X_6 = X_m^A \cap X_r^B$:

$$\begin{aligned} f_1(x) &= \min\big(\vec{T}(1,1), \vec{T}(1, \mathrm{ATM}(B)(x))\big) = \mathrm{ATM}(B)(x) \\ f_2(x) &= \vec{T}(1, \mathrm{ATM}(B)(x)) = \mathrm{ATM}(B)(x) \end{aligned}$$

7. $x \in X_7 = X_r^A \cap X_l^B$:

$$\begin{aligned} f_1(x) &= \min\big(\vec{T}(1, \mathrm{ATL}(B)(x)), \vec{T}(\mathrm{ATM}(A)(x), 1)\big) \\ &= \min(\mathrm{ATM}(A)(x), \mathrm{ATL}(B)(x)) \\ f_2(x) &= \vec{T}(\mathrm{ATM}(A)(x), \mathrm{ATL}(B)(x)) \end{aligned}$$

8. $x \in X_8 = X_r^A \cap X_m^B$:

$$\begin{aligned} f_1(x) &= \min\big(\vec{T}(1,1), \vec{T}(\mathrm{ATM}(A)(x), 1)\big) = \mathrm{ATM}(A)(x) \\ f_2(x) &= \vec{T}(\mathrm{ATM}(A)(x), 1) = \mathrm{ATM}(A)(x) \end{aligned}$$

9. $x \in X_9 = X_r^A \cap X_r^B$:

$$\begin{aligned} f_1(x) &= \min\big(\overleftrightarrow{T}(1,1), \overleftrightarrow{T}(\mathrm{ATM}(A)(x), \mathrm{ATM}(B)(x))\big) \\ &= \overleftrightarrow{T}(\mathrm{ATM}(A)(x), \mathrm{ATM}(B)(x)) \\ f_2(x) &= \overleftrightarrow{T}(\mathrm{ATM}(A)(x), \mathrm{ATM}(B)(x)) \end{aligned}$$

We see that, except on $X_3 \cup X_7$, the two functions f_1 and f_2 coincide. If $X_3 = X_7 = \emptyset$, (17) follows trivially, and we are done. So, to complete the proof, assume that $X_3 \neq \emptyset$ or $X_7 \neq \emptyset$. Let us consider the following two cases:

1. $X_3 \neq \emptyset$: then we can choose a $y' \in X_3 = X_l^A \cap X_r^B$. By Lemma II.22 we know that, for all $x' \in X_l^B \cup X_m^B$ and all $z' \in X_m^A \cup X_r^A$, the following must hold:

$$L(x', y') = 1 \qquad L(y', x') < 1 \qquad L(y', z') = 1 \qquad L(z', y') < 1 \tag{18}$$

   We now prove that

$$\begin{aligned} (X_m^A \cup X_r^A) &\cap (X_l^B \cup X_m^B) \\ &= (X_m^A \cap X_l^B) \cup (X_m^A \cap X_m^B) \cup (X_r^A \cap X_l^B) \cup (X_r^A \cap X_m^B) \\ &= X_4 \cup X_5 \cup X_7 \cup X_8 \end{aligned}$$

   is empty. Assume that there was an $x'' \in (X_m^A \cup X_r^A) \cap (X_l^B \cup X_m^B)$, then we can apply (18) with $x' = z' = x''$ and we obtain

$$L(x'', y') = 1, \qquad L(y', x'') < 1, \qquad L(y', x'') = 1, \qquad L(x'', y') < 1,$$

   which is a contradiction. Since $X_l^B = X_1 \cup X_4 \cup X_7$, we can further infer $X_l^B = X_1$. Analogously, the equalities $X_m^B = X_2$, $X_m^A = X_6$ and $X_r^A = X_9$ follow.

2. $X_7 \neq \emptyset$: analogously to the case $X_3 \neq \emptyset$, we can prove that

$$(X_l^A \cup X_m^A) \cap (X_m^B \cup X_r^B) = X_2 \cup X_3 \cup X_5 \cup X_6$$

   is empty and that the equalities $X_l^A = X_1$, $X_m^A = X_4$, $X_m^B = X_8$ and $X_r^B = X_9$ hold.

As a direct consequence, we obtain that the cases $X_3 \neq \emptyset$ and $X_7 \neq \emptyset$ are mutually exclusive.

In the following, let us assume that $X_3 \neq \emptyset$ (which implies $X_7 = \emptyset$). Since $X_4 \cup X_5 \cup X_7 \cup X_8 = \emptyset$, we have

$$\inf_{x \in X} f_1(x) = \min\begin{pmatrix} \inf\limits_{x \in X_1} f_1(x) \\ \inf\limits_{x \in X_2} f_1(x) \\ \inf\limits_{x \in X_3} f_1(x) \\ \inf\limits_{x \in X_6} f_1(x) \\ \inf\limits_{x \in X_9} f_1(x) \end{pmatrix}, \qquad \inf_{x \in X} f_2(x) = \min\begin{pmatrix} \inf\limits_{x \in X_1} f_2(x) \\ \inf\limits_{x \in X_2} f_2(x) \\ \inf\limits_{x \in X_3} f_2(x) \\ \inf\limits_{x \in X_6} f_2(x) \\ \inf\limits_{x \in X_9} f_2(x), \end{pmatrix}$$

where the above representations ensure that these two minima coincide except for their third arguments (the infima over X_3).

As the equalities

$$\begin{aligned}\inf_{x\in X_3} f_1(x) &= \inf_{x\in X_3} \min(\mathrm{ATL}(A)(x), \mathrm{ATM}(B)(x))\\ &= \min\big(\inf_{x\in X_3} \mathrm{ATL}(A)(x), \inf_{x\in X_3} \mathrm{ATM}(B)(x)\big)\end{aligned}$$

hold, we obtain the following:

$$\inf_{x\in X} f_1(x) = \min\begin{pmatrix} \inf_{x\in X_1} f_1(x)\\ \inf_{x\in X_2} f_1(x)\\ \inf_{x\in X_3} \mathrm{ATL}(A)(x)\\ \inf_{x\in X_3} \mathrm{ATM}(B)(x)\\ \inf_{x\in X_6} f_1(x)\\ \inf_{x\in X_9} f_1(x)\end{pmatrix} \tag{19}$$

We now prove the following lemma:

$$\begin{aligned}\inf_{x\in X_3} f_2(x) = \min\big(&\vec{T}(\sup_{x\in X_3} \mathrm{ATL}(A)(x), \inf_{x\in X_3} \mathrm{ATM}(B)(x)),\\ &\vec{T}(\sup_{x\in X_3} \mathrm{ATM}(B)(x), \inf_{x\in X_3} \mathrm{ATL}(A)(x))\big)\end{aligned} \tag{20}$$

To do so, consider the following inequality (using that $\vec{T}$ is non-increasing the first and non-decreasing in the second component):

$$\begin{aligned}\inf_{x\in X_3} f_2(x) &= \inf_{x\in X_3} \vec{T}(\mathrm{ATL}(A)(x), \mathrm{ATM}(B)(x))\\ &= \inf_{x\in X_3} \min\big(\vec{T}(\mathrm{ATL}(A)(x), \mathrm{ATM}(B)(x)), \vec{T}(\mathrm{ATM}(B)(x), \mathrm{ATL}(A)(x))\big)\\ &\geq \min\big(\vec{T}(\sup_{x\in X_3} \mathrm{ATL}(A)(x), \inf_{x\in X_3} \mathrm{ATM}(B)(x)),\\ &\qquad\quad \vec{T}(\sup_{x\in X_3} \mathrm{ATM}(B)(x), \inf_{x\in X_3} \mathrm{ATL}(A)(x))\big)\end{aligned}$$

Using that $\vec{T}$ is left-continuous in the first and right-continuous in the second component, we have

$$\vec{T}(\sup_{x\in X_3} \mathrm{ATL}(A)(x), \inf_{x\in X_3} \mathrm{ATM}(B)(x)) = \inf_{x_A, x_B\in X_3} \vec{T}(\mathrm{ATL}(A)(x_A), \mathrm{ATM}(B)(x_B)).$$

If we denote this value by r_1, then, for every $\varepsilon > 0$, we can choose two elements $x_A, x_B \in X_3$ such that

$$\vec{T}(\mathrm{ATL}(A)(x_A), \mathrm{ATM}(B)(x_B)) < r_1 + \varepsilon$$

holds. Since L is strongly complete, $L(x_A, x_B) = 1$ or $L(x_B, x_A) = 1$ holds. In the former case, $\mathrm{ATL}(A)(x_A) \leq \mathrm{ATL}(A)(x_B)$ holds by Lemma II.21, and we can infer

$$\begin{aligned}\vec{T}(\mathrm{ATL}(A)(x_B), \mathrm{ATM}(B)(x_B)) &\leq \vec{T}(\mathrm{ATL}(A)(x_B), \mathrm{ATM}(B)(x_B))\\ &\leq \vec{T}(\mathrm{ATL}(A)(x_A), \mathrm{ATM}(B)(x_B)) < r_1 + \varepsilon.\end{aligned}$$

In the case $L(x_B, x_A) = 1$, $\mathrm{ATM}(B)(x_A) \leq \mathrm{ATM}(B)(x_B)$ holds, and we obtain

$$\begin{aligned}\vec{T}(\mathrm{ATL}(A)(x_A), \mathrm{ATM}(B)(x_A)) &\leq \vec{T}(\mathrm{ATL}(A)(x_A), \mathrm{ATM}(B)(x_A)) \\ &\leq \vec{T}(\mathrm{ATL}(A)(x_A), \mathrm{ATM}(B)(x_B)) < r_1 + \varepsilon.\end{aligned}$$

This means that, for any $\varepsilon > 0$, we can always choose an $x \in X_3$ such that $f_2(x) \leq r_1 + \varepsilon$. Hence, we have $\inf_{x\in X_3} f_2(x) \leq r_1$. Analogously, we can prove that we can always find an $x \in X_3$ such that $f_2(x) \leq r_2 + \varepsilon$, where

$$r_2 = \vec{T}(\sup_{x\in X_3} \mathrm{ATM}(B)(x), \inf_{x\in X_3} \mathrm{ATL}(A)(x)).$$

Hence, we have $\inf_{x\in X_3} f_2(x) \leq r_2$ and we can conclude that

$$\inf_{x\in X_3} f_2(x) \leq \min(r_1, r_2),$$

which completes the proof of (20).

Using (20), we can infer the following:

$$\inf_{x\in X} f_2(x) = \min \begin{pmatrix} \inf_{x\in X_1} f_2(x) \\ \inf_{x\in X_2} f_2(x) \\ \vec{T}(\sup_{x\in X_3} \mathrm{ATM}(B)(x), \inf_{x\in X_3} \mathrm{ATL}(A)(x)) \\ \vec{T}(\sup_{x\in X_3} \mathrm{ATL}(A)(x), \inf_{x\in X_3} \mathrm{ATM}(B)(x))) \\ \inf_{x\in X_6} f_2(x) \\ \inf_{x\in X_9} f_2(x) \end{pmatrix} \tag{21}$$

To finally prove (17), let us first consider the case $X_2 = X_m^B \neq \emptyset$. Since $L(x, y) = 1$ for all $x \in X_m^B$ and all $y \in X_r^B$, we can infer that $\mathrm{ATL}(A)(x) \leq \mathrm{ATL}(A)(y)$ for all $x \in X_2 = X_m^B$ and all $y \in X_3 \subseteq X_r^B$. Thus,

$$\inf_{x\in X_2} f_1(x) = \inf_{x\in X_2} \mathrm{ATL}(A)(x) \leq \inf_{x\in X_3} \mathrm{ATL}(A)(x),$$

follows. Moreover, we have

$$\begin{aligned}\inf_{x\in X_2} f_2(x) &= \inf_{x\in X_2} \mathrm{ATL}(A)(x) \\ &\leq \inf_{x\in X_3} \mathrm{ATL}(A)(x) \\ &= \vec{T}(1, \inf_{x\in X_3} \mathrm{ATL}(A)(x)) \\ &\leq \vec{T}(\sup_{x\in X_3} \mathrm{ATM}(B)(x), \inf_{x\in X_3} \mathrm{ATL}(A)(x))\end{aligned}$$

So, for the case $X_2 = X_m^B \neq \emptyset$, we obtain that the third arguments of the minima in (19) and (21) are irrelevant.

Now consider the case $X_2 = X_m^B = \emptyset$. By Lemma II.23, $B \in \mathcal{F}_T(X)$ is a sufficient condition for $X_m^B \neq \emptyset$ to hold, i.e. $X_m^B = \emptyset$ implies $B \notin \mathcal{F}_T(X)$. We have assumed that $A, B \in \mathcal{F}_T(X) \cup \mathcal{F}_H(X)$; hence $X_m^B = \emptyset$ implies $B \in \mathcal{F}_H(X)$, i.e. $\mathrm{height}(B) = 1$ holds and we can infer

$$\begin{aligned} 1 = \mathrm{height}(B) &= \mathrm{height}(\mathrm{ECX}(B)) \\ &= \max\big(\sup_{x \in X_l^B} \mathrm{ECX}(B)(x), \sup_{x \in X_r^B} \mathrm{ECX}(B)(x)\big) \\ &= \max\big(\sup_{x \in X_l^B} \mathrm{ATL}(B)(x), \sup_{x \in X_r^B} \mathrm{ATM}(B)(x)\big). \end{aligned}$$

We know that $X_r^B = X_3 \cup X_6 \cup X_9$. Since $X_3 \subseteq X_l^A$, $X_6 \subseteq X_m^A$, and $X_9 \subseteq X_r^A$, we can infer that $L(x, y) = 1$ holds for all $x \in X_3$ and all $y \in X_6 \cup X_9$. Thus $\mathrm{ATM}(B)(x) \geq \mathrm{ATM}(B)(y)$ holds for $x \in X_3$ and all $y \in X_6 \cup X_9$. That is why we have

$$\sup_{x \in X_r^B} \mathrm{ATM}(B)(x) = \sup_{x \in X_3} \mathrm{ATM}(B)(x).$$

So we obtain (also using that $X_1 = X_l^B$)

$$\max\big(\sup_{x \in X_1} \mathrm{ATL}(B)(x), \sup_{x \in X_3} \mathrm{ATM}(B)(x)\big) = 1,$$

i.e. one of the two suprema must be 1. If we assume $\sup_{x \in X_1} \mathrm{ATL}(B)(x) = 1$, we can infer

$$\begin{aligned} \inf_{x \in X_1} f_1(x) &= \inf_{x \in X_1} \vec{T}(\mathrm{ATL}(A)(x), \mathrm{ATL}(B)(x)) \\ &\leq \inf_{x \in X_1} \vec{T}(\mathrm{ATL}(B)(x), \mathrm{ATL}(A)(x)) = (*). \end{aligned}$$

For all $x \in X_1 = X_l^B$ and all $y \in X_3 \subseteq X_r^B$, we have $L(x, y) = 1$ which implies $\mathrm{ATL}(A)(x) \leq \mathrm{ATL}(A)(y)$ by Lemma II.21. For an arbitrary $y \in X_3$, therefore, we can infer

$$\begin{aligned} (*) &\leq \inf_{x \in X_1} \vec{T}(\mathrm{ATL}(B)(x), \mathrm{ATL}(A)(y)) \\ &= \vec{T}(\sup_{x \in X_1} \mathrm{ATL}(B)(x), \mathrm{ATL}(A)(y)) \\ &= \vec{T}(1, \mathrm{ATL}(A)(y)) = \mathrm{ATL}(A)(y). \end{aligned}$$

This holds for all $y \in X_3$, hence

$$\inf_{x \in X_1} f_1(x) \leq \inf_{x \in X_3} \mathrm{ATL}(A)(x).$$

Moreover, we can further infer the following (the last inequality follows from the fact that $\vec{T}$ is non-increasing in the first argument):

$$\begin{aligned} \inf_{x \in X_1} f_2(x) &= \inf_{x \in X_1} f_1(x) \\ &\leq \inf_{x \in X_3} \mathrm{ATL}(A)(x) = \vec{T}(1, \inf_{x \in X_3} \mathrm{ATL}(A)(x)) \\ &= \vec{T}(\sup_{x \in X_3} \mathrm{ATM}(B)(x), \inf_{x \in X_3} \mathrm{ATL}(A)(x)) \end{aligned}$$

Now conversely assume that $\sup_{x\in X_3} \mathrm{ATM}(B)(x) = 1$. Then we immediately have

$$\begin{aligned}\vec{T}(\sup_{x\in X_3} \mathrm{ATM}(B)(x), \inf_{x\in X_3} \mathrm{ATL}(A)(x)) &= \vec{T}(1, \inf_{x\in X_3} \mathrm{ATL}(A)(x)) \\ &= \inf_{x\in X_3} \mathrm{ATL}(A)(x).\end{aligned}$$

Now let us summarize these findings:

1. In case that $X_m^B \neq \emptyset$ or that $\sup_{x\in X_1} \mathrm{ATL}(B)(x) = 1$, we obtain the following:

$$\left.\begin{array}{c}\inf\limits_{x\in X_1\cup X_2} f_1(x) \\ = \\ \inf\limits_{x\in X_1\cup X_2} f_2(x)\end{array}\right\} \leq \left\{\begin{array}{l}\inf\limits_{x\in X_3} \mathrm{ATL}(A)(x) \\ \vec{T}(\sup\limits_{x\in X_3} \mathrm{ATM}(B)(x), \inf\limits_{x\in X_3} \mathrm{ATL}(A)(x))\end{array}\right.$$

2. In case that $\sup_{x\in X_3} \mathrm{ATM}(B)(x) = 1$, we obtain

$$\vec{T}(\sup_{x\in X_3} \mathrm{ATM}(B)(x), \inf_{x\in X_3} \mathrm{ATL}(A)(x)) = \inf_{x\in X_3} \mathrm{ATL}(A)(x).$$

Hence we can infer the following:

$$\min\begin{pmatrix}\inf\limits_{x\in X_1} f_1(x) \\ \inf\limits_{x\in X_2} f_1(x) \\ \inf\limits_{x\in X_3} \mathrm{ATL}(A)(x)\end{pmatrix} = \min\begin{pmatrix}\inf\limits_{x\in X_1} f_2(x) \\ \inf\limits_{x\in X_2} f_2(x) \\ \vec{T}(\sup\limits_{x\in X_3} \mathrm{ATM}(B)(x), \inf\limits_{x\in X_3} \mathrm{ATL}(A)(x))\end{pmatrix} \tag{22}$$

Analogously to above, we can prove that the following holds in the case that $X_6 = X_m^A \neq \emptyset$ or $\sup_{x\in X_9} \mathrm{ATM}(A)(x) = 1$:

$$\left.\begin{array}{c}\inf\limits_{x\in X_6\cup X_9} f_1(x) \\ = \\ \inf\limits_{x\in X_6\cup X_9} f_2(x)\end{array}\right\} \leq \left\{\begin{array}{l}\inf\limits_{x\in X_3} \mathrm{ATM}(B)(x) \\ \vec{T}(\sup\limits_{x\in X_3} \mathrm{ATL}(A)(x), \inf\limits_{x\in X_3} \mathrm{ATM}(B)(x))\end{array}\right.$$

In case that $\sup_{x\in X_3} \mathrm{ATL}(A)(x) = 1$, we can infer

$$\vec{T}(\sup_{x\in X_3} \mathrm{ATL}(A)(x), \inf_{x\in X_3} \mathrm{ATM}(B)(x)) = \inf_{x\in X_3} \mathrm{ATM}(B)(x).$$

Finally, we obtain the following:

$$\min\begin{pmatrix}\inf\limits_{x\in X_3} \mathrm{ATM}(B)(x) \\ \inf\limits_{x\in X_6} f_1(x) \\ \inf\limits_{x\in X_9} f_1(x)\end{pmatrix} = \min\begin{pmatrix}\vec{T}(\sup\limits_{x\in X_3} \mathrm{ATL}(A)(x), \inf\limits_{x\in X_3} \mathrm{ATM}(B)(x)) \\ \inf\limits_{x\in X_6} f_2(x) \\ \inf\limits_{x\in X_9} f_2(x)\end{pmatrix} \tag{23}$$

Putting (22) and (23) together and taking (19) and (21) into account, (17) follows finally. The proof for the case $X_7 \neq \emptyset$ (and $X_3 = \emptyset$) can be carried out analogously just by swapping A and B. □

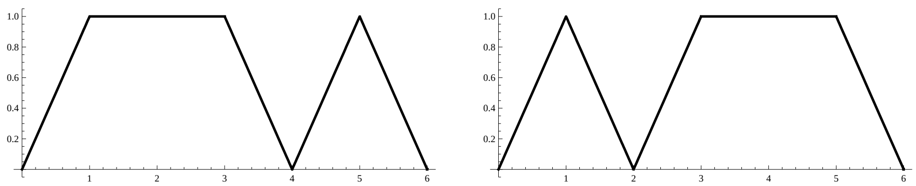

Figure 5: Two non-convex fuzzy quantities A_8 (left) and B_8 (right) that have equal convex hulls.

Finally, let us check which results we obtain if we apply the fuzzy relation $\mathcal{L}'_L$ to the examples considered so far.

Example II.17. Consider the same setting as in Example II.5, i.e. we use $T_{\mathbf{L}}$ as underlying t-norm and define L to be the crisp linear ordering of real numbers. The fuzzy quantities $A_1, \ldots, A_4$ and $B_1, \ldots, B_4$ were all normal, so $\mathcal{L}'_L$ gives the same results as $\mathcal{L}_L$. For the fuzzy quantities from Example II.6, we obtain the following:

$$\begin{aligned} \mathcal{L}'_L(A_5, B_5) &= 1 & \qquad \mathcal{L}'_L(B_5, A_5) &= 0 \\ \mathcal{L}'_L(A_6, B_6) &= 0.5625 & \qquad \mathcal{L}'_L(B_6, A_6) &= 0.375 \end{aligned}$$

So the limitations of $\mathcal{L}_L$ are overcome for the pair (A_5, B_5), which is not surprising if we look at Example II.15. For the pair (A_6, B_6), we obtain the same results as for $\mathcal{L}_L$ in Example II.6.

For L as defined in Example II.14 and the two fuzzy quantities A_7 and B_7, we obtain

$$\mathcal{L}'_L(A_7, B_7) = \mathcal{L}'_L(B_7, A_7) = 0.6.$$

As $\lceil \mathrm{ECX}(A_7) \rceil = \lceil \mathrm{ECX}(B_7) \rceil$ holds, we have $\mathrm{SIM}_{T_{\mathbf{L}}}(\lceil \mathrm{ECX}(A_7) \rceil, \lceil \mathrm{ECX}(B_7) \rceil) = 1$, and we see that (16) is not guaranteed to be fulfilled if L is not strongly complete.

II.5 Hybridization by Lexicographic Refinement

If we assume that there are applications in which it is not appropriate to consider only (extensional) convex fuzzy sets, the above approach—no matter whether a crisp or a fuzzy variant is considered—can run into problems, simply because of its inability to distinguish between fuzzy alternatives with equal (extensional) convex hulls. As already mentioned in Subsection I.4.3, the two fuzzy quantities in Figure 5 (let us call them A_8 and B_8 from now on) demonstrate, that the results concerning the antisymmetry of the preordering $\preceq_L$ are not necessarily exhaustive.

One possible way to gain "more antisymmetry" while keeping all our present achievements is hybridization with some other ordering method by lexicographic refinement. As usual, this means that, if one ordering cannot distinguish between two alternatives, we use a second one. For a real-world analogue, consider ordering entries in a dictionary or phone book: if the first letters of two entries are the same (i.e. the first letter is not sufficient to put them in proper order), the second letter is used to order them, and so forth.

In the following, we describe this process in detail for the preordering $\preceq_L$ (for some choice of L; the construction works in the same way for $\preceq'_L$). Assume that we have another preordering of fuzzy sets $\preceq''$ and that $\mathcal{S} \subseteq \mathcal{F}(X)$ is the sub-class of fuzzy sets on X for which $\preceq''$ is applicable. Then the relation

$$A \precsim_L B \ \Leftrightarrow \big((A \cong_L B \,\&\, A \preceq'' B) \vee (A \preceq_L B \,\&\, B \not\preceq_L A)\big)$$

is a preordering on $\mathcal{S}$, where the following properties hold obviously:

1. If $\preceq''$ is an ordering, $\precsim_L$ is an ordering.
2. The relation $\precsim_L$ is a sub-relation of $\preceq_L$´, i.e. $A \precsim_L B$ implies $A \preceq_L B$ for all $A, B \in \mathcal{S}$. This also entails that the original preordering $\preceq_L$ has priority over $\preceq''$ in the sense that $A \not\preceq_L B$ implies $A \not\precsim_L B$
3. The symmetric kernel of $\precsim_L$ is the intersection of the symmetric kernels of the two relations, i.e. for all $A, B \in \mathcal{S}$,
$$(A \precsim_L B \,\&\, B \precsim_L A) \ \Leftrightarrow \ (A \cong_L B \,\&\, A \preceq'' B \,\&\, B \preceq'' A), \tag{24}$$
which means that $\precsim_L$ is "at least as antisymmetric" as $\preceq_L$ and $\preceq''$.

There is, however, one important aspect that should not be overlooked. If $\preceq$ is not linear, we can come to the peculiar situation that two fuzzy sets are treated as equal by $\preceq_L$ but incomparable with respect to $\precsim_L$.

In the case of the real numbers $X = \mathbb{R}$, a possibility is to use an existing ordering method that is reflexive and transitive. The simplest case could be a method that orders fuzzy quantities by one characteristic value (methods of the first class according to Wang's and Kerre's classification [25,26]). Of course, mapping a fuzzy set to one single value results in a dramatic loss of information as already pointed out by Freeling [16], Wang, and Kerre [25,26]. However, the influence of this loss is, in this specific case, limited, because we are only considering fuzzy quantities with equal (extensional) convex hulls. The only crucial thing is whether the method can yield an improvement at all. Adamo's method of considering the rightmost value of some α-cut [1], for instance, is even less specific than the relation $\preceq_I$. The same happens with the approach of Fortemps and Roubens which also considers just the boundaries of α-cuts [15], and other methods which are only applicable for convex fuzzy quantities [12,22]. Thinking of the example in Figure 5, comparing the centers of gravity of the membership functions [11] would do a perfect job, the Yager indices [27] as well.

Example II.18. Define $\mathcal{S}$ to be the set of fuzzy quantities with integrable membership function. Now define

$$A \preceq'' B \ \Leftrightarrow \ \mathrm{COG}(A) \leq \mathrm{COG}(B),$$

where COG denotes the well-known center of gravity defuzzification. If we let L again be the crisp linear ordering of real numbers, we can directly infer that $A_8 \precsim_L B_8$ holds, while $B_8 \not\precsim_L A_8$, i.e. the refinement of $\preceq_L$ by means of the ordering of centers of gravity improves the situation in this case. Note that $\preceq''$ is a linear ordering, so the peculiar situation mentioned above cannot occur.

Beside the above approaches, there are a lot of other methods for ordering fuzzy quantities. It is beyond the scope of this paper to check the properties of all possible combinations—the demands on such a method depend on the specific application anyway.

Finally, it is worth to mention that there is neither a theoretical nor a practical obstacle to repeat lexicographic refinement once or even more often. This iterative process always yields a preordering as long as all relations involved are preorderings.

The question remains how lexicographic refinement can be done for the fuzzy orderings of fuzzy sets discussed above. The following theorem provides us with a means to accomplish that.

Theorem II.19. *Consider two T-equivalences $E_1 : X^2 \to [0,1]$, $E_2 : X^2 \to [0,1]$, a T-E_1-ordering $L_1 : X^2 \to [0,1]$, and a T-E_2-ordering $L_2 : X^2 \to [0,1]$. Moreover, let $\tilde{T}$ be a t-norm that dominates T. Then the fuzzy relation $\mathrm{Lex}_{\tilde{T},T}(L_1, L_2) : X^2 \to [0,1]$ defined as*

$$\mathrm{Lex}_{\tilde{T},T}(L_1,L_2)(x,y) = \max\big(\tilde{T}(L_1(x,y), L_2(x,y)), \min(L_1(x,y), N_T(L_1(y,x)))\big)$$

is a fuzzy ordering with respect to T and the $\tilde{T}$-intersection of E_1 and E_2:

$$\mathrm{Int}_{\tilde{T}}(E_1,E_2)(x,y) = \tilde{T}(E_1(x,y), E_2(x,y))$$

Proof. Analogous to [6, Theorem 11]. □

We can now apply Theorem II.19 directly to hybridize/refine the fuzzy ordering $\mathcal{L}_L$ (analogously for $\mathcal{L}'_L$) with some other fuzzy ordering of fuzzy sets.

Example II.20. Define $\mathcal{S}$ to be the set of fuzzy quantities with integrable membership function and let L be the crisp linear ordering of real numbers again. Now we define a binary fuzzy relation $\mathcal{L}'' : \mathcal{S}^2 \to [0,1]$ as follows:

$$\mathcal{L}''(A,B) = \max(\min(1 - \mathrm{COG}(A) + \mathrm{COG}(B), 1, 0)$$

Note that this is nothing else but the fuzzy ordering from Example I.10 applied to centers of gravities. Hence, it follows that $\mathcal{L}''$ is a fuzzy ordering with respect to the Łukasiewicz t-norm $T_{\mathbf{L}}$ and the following $T_{\mathbf{L}}$-equivalence:

$$\mathcal{E}''(A,B) = \max(1 - |\,\mathrm{COG}(A) - \mathrm{COG}(B)|, 0)$$

Then, using $T = T_{\mathbf{L}}$ and choosing $\tilde{T}$ such that it dominates $T_{\mathbf{L}}$, Theorem II.19 implies that[3]

$$\begin{aligned}\mathrm{Lex}_{\tilde{T},T_{\mathbf{L}}}(\mathcal{L}_L,\mathcal{L}'')(A,B) = \max\big(&\tilde{T}(\mathcal{L}_L(A,B), \mathcal{L}''(A,B)),\\ &\min(\mathcal{L}_L(A,B), 1 - \mathcal{L}_L(B,A))\big)\end{aligned}$$

is a fuzzy ordering on $\mathcal{S}$ with respect to $T_{\mathbf{L}}$ and the $T_{\mathbf{L}}$-equivalence

$$\mathrm{Int}_{\tilde{T}}(\mathcal{E}_L,\mathcal{E}'')(A,B) = \tilde{T}(\mathcal{E}_L(A,B), \mathcal{E}''(A,B)).$$

[3] Note that $N_{T_{\mathbf{L}}}(x) = \min(1 - x + 0, 1) = 1 - x$ holds.

The fact that the lexicographic refinement is antisymmetric with respect to the intersection $\mathrm{Int}_{\tilde{T}}(\mathcal{E}_L, \mathcal{E}'')$ is a perfect analogue to the crisp antisymmetry (24).

For (A_1, B_1), we know from Example II.5 that we have $\mathcal{L}_L(A_1, B_1) = 1$ and $\mathcal{L}_L(B_1, A_1) = \mathcal{E}_L(A_1, B_1) = 0$. It is easy to see that $\mathrm{COG}(A_1) = 1$ and $\mathrm{COG}(B_1) = 2.3$ hold. Thus, we obtain $\mathcal{L}''(A_1, B_1) = 1$ and $\mathcal{L}''(B_1, A_1) = \mathcal{E}''(A_1, B_1) = 0$. So this is a clear case and we obtain

$$\mathrm{Lex}_{\tilde{T},T_{\mathbf{L}}}(\mathcal{L}_L, \mathcal{L}'')(A_1, B_1) = \max\big(\tilde{T}(1,1), \min(1, 1-0)\big) = 1,$$
$$\mathrm{Lex}_{\tilde{T},T_{\mathbf{L}}}(\mathcal{L}_L, \mathcal{L}'')(B_1, A_1) = \max\big(\tilde{T}(0,0), \min(0, 1-1)\big) = 0,$$

and $\mathrm{Int}_{\tilde{T}}(\mathcal{E}_L, \mathcal{E}'')(A_1, B_1) = 0$ regardless of which $\tilde{T}$ we have chosen.

For the example (A_2, B_2), we have $\mathrm{COG}(A_2) = 1.0177$ and $\mathrm{COG}(B_2) = 2.1672$, and the computations and results are exactly the same as for the pair (A_1, B_1).

For (A_3, B_3), we obtain (using $\mathrm{COG}(A_3) = 1$ and $\mathrm{COG}(B_3) = 2.3$)

$$\mathrm{Lex}_{\tilde{T},T_{\mathbf{L}}}(\mathcal{L}_L, \mathcal{L}'')(A_3, B_3) = \max\big(\tilde{T}(0.9,1), \min(0.9, 1-0)\big) = 0.9,$$
$$\mathrm{Lex}_{\tilde{T},T_{\mathbf{L}}}(\mathcal{L}_L, \mathcal{L}'')(B_3, A_3) = \max\big(\tilde{T}(0,0), \min(0, 1-1)\big) = 0,$$

which implies $\mathrm{Int}_{\tilde{T}}(\mathcal{E}_L, \mathcal{E}'')(A_3, B_3) = 0$ (again regardless of the choice of $\tilde{T}$).

For (A_4, B_4), we have $\mathrm{COG}(A_4) = 2.4\dot{3}$ and $\mathrm{COG}(B_4) = 2.5$, and we obtain

$$\mathcal{L}''(A_4, B_4) = 1, \qquad \mathcal{L}''(B_4, A_4) = 0.9\dot{3}, \qquad \mathcal{E}''(A_4, B_4) = 0.9\dot{3},$$

which implies the following:

$$\begin{aligned}
\mathrm{Lex}_{\tilde{T},T_{\mathbf{L}}}(\mathcal{L}_L, \mathcal{L}'')(A_4, B_4) &= \max\big(\tilde{T}(0.625, 1), \min(0.625, 0.375)\big) = 0.625\\
\mathrm{Lex}_{\tilde{T},T_{\mathbf{L}}}(\mathcal{L}_L, \mathcal{L}'')(B_4, A_4) &= \max\big(\tilde{T}(0.41\dot{6}, 0.9\dot{3}), \min(0.41\dot{6}, 0.58\dot{3})\big)\\
&= \tilde{T}(0.41\dot{6}, 0.9\dot{3})\\
\mathrm{Int}_{\tilde{T}}(\mathcal{E}_L, \mathcal{E}'')(A_4, B_4) &= \tilde{T}(0.625, 0.41\dot{6}, 0.9\dot{3})
\end{aligned}$$

We see that the results for (A_4, B_4) are dependent of the choice of $\tilde{T}$. For the two cases $\tilde{T} = T_{\mathbf{M}}$ and $\tilde{T} = T_{\mathbf{L}}$, we obtain the following results:

$$\begin{aligned}
\mathrm{Lex}_{T_{\mathbf{M}},T_{\mathbf{L}}}(\mathcal{L}_L, \mathcal{L}'')(A_4, B_4) &= 0.625 & \mathrm{Lex}_{T_{\mathbf{L}},T_{\mathbf{L}}}(\mathcal{L}_L, \mathcal{L}'')(A_4, B_4) &= 0.625\\
\mathrm{Lex}_{T_{\mathbf{M}},T_{\mathbf{L}}}(\mathcal{L}_L, \mathcal{L}'')(B_4, A_4) &= 0.41\dot{6} & \mathrm{Lex}_{T_{\mathbf{L}},T_{\mathbf{L}}}(\mathcal{L}_L, \mathcal{L}'')(B_4, A_4) &= 0.35\\
\mathrm{Int}_{T_{\mathbf{M}}}(\mathcal{L}_L, \mathcal{L}'')(A_4, B_4) &= 0.41\dot{6} & \mathrm{Int}_{T_{\mathbf{L}}}(\mathcal{L}_L, \mathcal{L}'')(A_4, B_4) &= 0
\end{aligned}$$

The case (A_5, B_5) is analogous to the example (A_3, B_3), and the case (A_6, B_6) is analogous to the example (A_4, B_4).

For the example (A_8, B_8), we have $\mathrm{COG}(A_8) = 2.75$ and $\mathrm{COG}(B_8) = 3.25$. So we have the following:

$$\begin{aligned}
&\mathcal{L}_L(A_8, B_8) = 1 & &\mathcal{L}_L(B_8, A_8) = 1 & &\mathcal{E}_L(A_8, B_8) = 1\\
&\mathcal{L}''(A_8, B_8) = 1 & &\mathcal{L}''(B_8, A_8) = 0.5 & &\mathcal{E}''(A_8, B_8) = 0.5
\end{aligned}$$

Hence, we obtain the following results (independent of the choice of $\tilde{T}$):

$$\begin{aligned}
\mathrm{Lex}_{\tilde{T},T_{\mathbf{L}}}(\mathcal{L}_L,\mathcal{L}'')(A_8,B_8) &= \max\big(\tilde{T}(1,1),\min(1,0)\big) = 1\\
\mathrm{Lex}_{\tilde{T},T_{\mathbf{L}}}(\mathcal{L}_L,\mathcal{L}'')(B_8,A_8) &= \max\big(\tilde{T}(1,0.5),\min(1,0)\big) = 0.5\\
\mathrm{Int}_{\tilde{T}}(\mathcal{E}_L,\mathcal{E}'')(A_8,B_8) &= 0.5
\end{aligned}$$

II.6 Concluding Remarks

In this paper, we have addressed the three limitations of the general fuzzy ordering-based ordering of fuzzy sets (as noted in Section I.4). By fuzzification using a fuzzy inclusion measure, we could overcome the problem of "artificial preciseness". The issue of incomparability caused by different heights has been solved using lifting. That the ordering relation is unable to distinguish between fuzzy sets with the same (extensional) convex hulls can be solved by lexicographic refinement with another ordering method. It is worth to point out that these three extensions are not mutually exclusive. Instead, the lifting approach is available in a crisp and in a fuzzy variant. Lexicographic refinement can be done both with the crisp and the fuzzy variant and regardless of whether original or lifted variants are considered.

We are convinced that the framework introduced in this paper provides the reader and potential user with a tool chest from which he/she can choose an appropriate ordering method that is suitable for his/her concrete application.

Appendix: Helpful Lemmata

In this section, we provide some useful lemmata that are of a more general nature and not directly related to the main focus of this paper. To our best knowledge, these lemmata have not been proven anywhere else in literature. That is why we provide them along with full proofs.

Lemma II.21. *Consider a T-equivalence $E : X^2 \to [0,1]$, a T-E-ordering $L : X^2 \to [0,1]$, and a fuzzy set $A \in \mathcal{F}(X)$. Then the following inequalities hold (for all $x, y \in X$):*

$$\begin{aligned}
L(x,y) &\leq \vec{T}(\mathrm{ATL}(A)(x),\mathrm{ATL}(A)(y))\\
L(x,y) &\leq \vec{T}(\mathrm{ATM}(A)(y),\mathrm{ATM}(A)(x))
\end{aligned}$$

As a consequence, $L(x,y) = 1$ implies the inequalities $\mathrm{ATL}(A)(x) \leq \mathrm{ATL}(A)(y)$ and $\mathrm{ATM}(A)(x) \geq \mathrm{ATM}(A)(y)$.

Proof. Consider the following chain of equalities (using the left-continuity of the t-norm T and the T-transitivity of L):

$$\begin{aligned}
T(\mathrm{ATL}(A)(x), L(x,y)) &= T\big(\sup_{z\in X} T(A(z), L(z,x)), L(x,y)\big) \\
&= \sup_{z\in X} T\big(A(z), L(z,x), L(x,y)\big) \\
&\leq \sup_{z\in X} T\big(A(z), L(z,y)\big) = \mathrm{ATL}(A)(y).
\end{aligned}$$

So we have proven $T(L(x,y), \mathrm{ATL}(A)(x)) \leq \mathrm{ATL}(A)(y)$ which is, by (I2), equivalent to

$$L(x,y) \leq \vec{T}(\mathrm{ATL}(A)(x), \mathrm{ATL}(A)(y)).$$

If $L(x,y) = 1$ holds, we can infer $\vec{T}(\mathrm{ATL}(A)(x), \mathrm{ATL}(A)(y)) = 1$ which is, by (I1), equivalent to $\mathrm{ATL}(A)(x) \leq \mathrm{ATL}(A)(y)$. The corresponding results for $\mathrm{ATM}(A)$ can be proven analogously. □

In other words, Lemma II.21 states that the membership function of $\mathrm{ATL}(A)$ is non-decreasing with respect to L and the membership function of $\mathrm{ATM}(A)$ is non-increasing with respect to L (in a graded sense à la [9]). The two final inequalities state that the membership functions of $\mathrm{ATL}(A)$ and $\mathrm{ATM}(A)$ are non-decreasing and non-increasing, respectively, with respect to the kernel relation of L [4, Lemma 4.1].

Lemma II.22. *Consider a T-equivalence $E : X^2 \to [0,1]$, a strongly complete T-E-ordering $L : X^2 \to [0,1]$, and a fuzzy set $A \in \mathcal{F}(X)$. Then there exist three sets $X_l, X_m, X_r \subseteq X$ that are a partition of X and have the following properties:*

1. *The (in)equalities*

$$\begin{array}{ll}
L(x,y) = 1 & \qquad L(y,x) < 1 \\
L(y,z) = 1 & \qquad L(z,y) < 1 \\
L(x,z) = 1 & \qquad L(z,x) < 1
\end{array}$$

   hold for all $x \in X_l$, all $y \in X_m$, and all $z \in X_r$.

2. *The following representations hold:*

$$\mathrm{ATL}(A)(x) = \begin{cases} \mathrm{ECX}(A)(x) & \text{if } x \in X_l \\ \mathrm{height}(A) & \text{otherwise} \end{cases}$$

$$\mathrm{ATM}(A)(x) = \begin{cases} \mathrm{ECX}(A)(x) & \text{if } x \in X_r \\ \mathrm{height}(A) & \text{otherwise} \end{cases}$$

$$\mathrm{ECX}(A)(x) = \begin{cases} \mathrm{ATL}(A)(x) & \text{if } x \in X_l \\ \mathrm{height}(A) & \text{if } x \in X_m \\ \mathrm{ATM}(A)(x) & \text{if } x \in X_r \end{cases}$$

3. Furthermore, the following equalities hold:

$$\begin{aligned}
\mathrm{ceiling}(\mathrm{ATL}(A)) &= X_m \cup X_r\\
\mathrm{ceiling}(\mathrm{ECX}(A)) &= X_m\\
\mathrm{ceiling}(\mathrm{ATM}(A)) &= X_l \cup X_m
\end{aligned}$$

Proof. We define:

$$\begin{aligned}
X_l &= \{x \in X \mid \mathrm{ATL}(A)(x) < \mathrm{ATM}(A)(x)\}\\
X_m &= \{x \in X \mid \mathrm{ATL}(A)(x) = \mathrm{ATM}(A)(x)\}\\
X_r &= \{x \in X \mid \mathrm{ATL}(A)(x) > \mathrm{ATM}(A)(x)\}
\end{aligned}$$

It is trivial that X_l, X_m, and X_r are forming a partition, as exactly one of the three defining properties must hold for every $x \in X$.

Since L is strongly complete, we have

$$\mathrm{height}(A) = \sup_{y\in X} A(y) = \max\Big(\sup_{L(y,x)=1} A(y), \sup_{L(x,y)=1} A(y)\Big) \tag{25}$$

for every $x \in X$. Now choose an arbitrary $x \in X_l$. Then

$$\begin{aligned}
\mathrm{height}(A) \geq \mathrm{ATM}(A)(x) > \mathrm{ATL}(A)(x) &= \sup_{y\in X} T(A(y), L(y,x))\\
&\geq \sup_{L(y,x)=1} T(A(y), L(y,x)) = \sup_{L(y,x)=1} A(y)
\end{aligned}$$

holds which, together with (25), implies

$$\mathrm{height}(A) = \sup_{L(x,y)=1} A(y)$$

for all $x \in X_l$. Now consider the following:

$$\begin{aligned}
\mathrm{ATM}(A)(x) &= \sup_{y\in X} T(A(y), L(x,y))\\
&= \max\Big(\sup_{L(x,y)=1} T(A(y), L(x,y)), \sup_{L(y,x)=1} T(A(y), L(x,y))\Big)\\
&\geq \sup_{L(x,y)=1} T(A(y), L(x,y)) = \sup_{L(x,y)=1} A(y)\\
&= \mathrm{height}(A)
\end{aligned}$$

So we have proven finally that $\mathrm{ATM}(A)(x) = \mathrm{height}(A)$ if $\mathrm{ATL}(A)(x) < \mathrm{ATM}(A)(x)$ (i.e. if $x \in X_l$). That $\mathrm{ATL}(A)(x) = \mathrm{ECX}(A)(x)$ holds in this case follows trivially from the definition $\mathrm{ECX}(A) = \mathrm{ATL}(A) \cap \mathrm{ATM}(A)$.

Analogously, we can prove that $\mathrm{ATL}(A)(x) = \mathrm{height}(A)$ and $\mathrm{ATM}(A)(x) = \mathrm{ECX}(A)(x)$ for all $x \in X_r$.

From the definition of $\mathrm{ECX}(A)$, we can infer trivially

$$\mathrm{ATL}(A)(x) = \mathrm{ATM}(A)(x) = \mathrm{ECX}(A)(x)$$

for all $x \in X_m$. Then we can infer

$$\begin{aligned}\mathrm{ECX}(A)(x) &= \max(\mathrm{ATL}(A)(x), \mathrm{ATM}(A)(x)) \\ &= \max\Big(\sup_{y\in X} T(A(y), L(y,x)), \sup_{y\in X} T(A(y), L(x,y))\Big) \\ &\geq \sup_{y\in X} T(A(y), \underbrace{\max(L(x,y), L(y,x))}_{=1}) = \mathrm{height}(A).\end{aligned}$$

Hence, $\mathrm{ATL}(A)(x) = \mathrm{ATM}(A)(x) = \mathrm{ECX}(A)(x) = \mathrm{height}(A)$ for all $x \in X_m$, and all representations of Part 2. are proven.

Now we prove Part 1. of the theorem. Choose an $x \in X_l$ and a $y \in X_m$. From 2., we know that

$$\mathrm{ATL}(A)(x) < \mathrm{ATL}(A)(y) = \mathrm{height}(A)$$

holds. By applying contraposition to Lemma II.21, this implies $L(y,x) < 1$, and $L(x,y) = 1$ follows from the strong completeness of L. The proof works in the same way if we replace $y \in X_m$ by a $z \in X_r$. The proof that $L(y,z) = 1$ holds for all $y \in X_m$ and all $z \in X_r$ can be done analogously using the fact that $\mathrm{ATM}(A)$ is non-increasing with respect to L.[4]

The representations of ceiling(ATL(A)), ceiling(ECX(A)) and ceiling(ATM(A)) follow trivially. □

Note that the requirement of strong completeness is essential in Lemma II.22. To see that, consider Example II.14: L is obviously not strongly complete and the two fuzzy quantities A_7 and B_7 both do not facilitate the representations of Lemma II.22.

Lemma II.23. *Consider a T-equivalence $E : X^2 \to [0,1]$, a strongly complete T-E-ordering $L : X^2 \to [0,1]$, and a fuzzy set $A \in \mathcal{F}(X)$. Let the three sets $X_l, X_m, X_r \subseteq X$ be defined as in Lemma II.22. Then $A \in \mathcal{F}_T(X)$, implies $X_m \neq \emptyset$.*

Proof. Assume that $A \in \mathcal{F}_T(X)$, i.e. there exists an $x \in X$ such that $A(x) = \mathrm{height}(A)$. Since $\mathrm{ATL}(A)$, $\mathrm{ECX}(A)$ and $\mathrm{ATM}(A)$ are fuzzy supersets of A while

$$\mathrm{height}(\mathrm{ATL}(A)) = \mathrm{height}(\mathrm{ECX}(A)) = \mathrm{height}(\mathrm{ATM}(A)) = \mathrm{height}(A)$$

holds by Lemma I.22, we can infer

$$\mathrm{ATL}(A)(x) = \mathrm{ECX}(A)(x) = \mathrm{ATM}(A)(x) = \mathrm{height}(A),$$

thus, by Lemma II.22, $x \in X_m$. □

[4]Note that up to two of the three sets X_l, X_m, X_r may by empty. If X_m was non-empty, the equality $L(x,z) = 1$ would follow from $L(x,y) = 1$ and $L(y,z) = 1$ by the T-transitivity of L. Since we cannot assume $X_m \neq \emptyset$ in general, we have to prove $L(x,z) = 1$ separately.

References

[1] J. M. Adamo. Fuzzy decision trees. *Fuzzy Sets and Systems*, 4:207–219, 1980.

[2] W. Bandler and L. J. Kohout. Fuzzy power sets and fuzzy implication operators. *Fuzzy Sets and Systems*, 4:183–190, 1980.

[3] U. Bodenhofer. *A Similarity-Based Generalization of Fuzzy Orderings*, volume C 26 of *Schriftenreihe der Johannes-Kepler-Universität Linz*. Universitätsverlag Rudolf Trauner, 1999.

[4] U. Bodenhofer. A similarity-based generalization of fuzzy orderings preserving the classical axioms. *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems*, 8(5):593–610, 2000.

[5] U. Bodenhofer. Representations and constructions of similarity-based fuzzy orderings. *Fuzzy Sets and Systems*, 137(1):113–136, 2003.

[6] U. Bodenhofer. Lexicographic composition of fuzzy orderings. In *Proc. 11th Int. Conf. on Information Processing and Management of Uncertainty in Knowledge-Based Systems*, volume 3, pages 2430–2437, Paris, July 2006.

[7] U. Bodenhofer. Orderings of fuzzy sets based on fuzzy orderings. part I: the basic approach. *Mathware Soft Comput.*, 15(2), 2008.

[8] U. Bodenhofer, M. De Cock, and E. E. Kerre. Openings and closures of fuzzy preorderings: theoretical basics and applications to fuzzy rule-based systems. *Int. J. General Systems*, 32(4):343–360, 2003.

[9] L. Běhounek, U. Bodenhofer, and P. Cintula. Relations in Fuzzy Class Theory: Initial steps. *Fuzzy Sets and Systems*, 159(14):1729–1772, 2008.

[10] L. Běhounek and P. Cintula. Fuzzy class theory. *Fuzzy Sets and Systems*, 154(1):34–55, 2005.

[11] W. Chang. Ranking of fuzzy utilities with triangular membership functions. In *Proc. Int. Conf. on Policy Anal. and Inf. Systems*, pages 263–272, 1981.

[12] F. Choobineh and H. Li. An index for ordering fuzzy numbers. *Fuzzy Sets and Systems*, 54:287–294, 1993.

[13] B. De Baets and R. Mesiar. T-partitions. *Fuzzy Sets and Systems*, 97:211–223, 1998.

[14] J. Fodor and M. Roubens. *Fuzzy Preference Modelling and Multicriteria Decision Support*. Kluwer Academic Publishers, Dordrecht, 1994.

[15] P. Fortemps and M. Roubens. Ranking and defuzzification methods based on area decomposition. *Fuzzy Sets and Systems*, 82:319–330, 1996.

[16] S. Freeling. Fuzzy sets and decision analysis. *IEEE Trans. Syst. Man Cybern.*, 10:341–354, 1980.

[17] S. Gottwald. *Fuzzy Sets and Fuzzy Logic*. Vieweg, Braunschweig, 1993.

[18] S. Gottwald. *A Treatise on Many-Valued Logics*. Studies in Logic and Computation. Research Studies Press, Baldock, 2001.

[19] P. Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer Academic Publishers, Dordrecht, 1998.

[20] E. P. Klement, R. Mesiar, and E. Pap. *Triangular Norms*, volume 8 of *Trends in Logic*. Kluwer Academic Publishers, Dordrecht, 2000.

[21] L. T. Kóczy. Personal communication, 1998.

[22] T. Liou and J. Wang. Ranking fuzzy numbers with integral value. *Fuzzy Sets and Systems*, 50:147–255, 1992.

[23] B. Moser. Personal communication, 1998.

[24] S. Saminger, R. Mesiar, and U. Bodenhofer. Domination of aggregation operators and preservation of transitivity. *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems*, 10(Suppl.):11–35, 2002.

[25] X. Wang and E. E. Kerre. Reasonable properties for the ordering of fuzzy quantities (I). *Fuzzy Sets and Systems*, 118:375–385, 2001.

[26] X. Wang and E. E. Kerre. Reasonable properties for the ordering of fuzzy quantities (II). *Fuzzy Sets and Systems*, 118:387–405, 2001.

[27] R. R. Yager. A procedure for ordering fuzzy subsets over the unit interval. *Inform. Sci.*, 24:143–161, 1981.